Exit Dilemma*

The Role of Private Learning on Firm Survival

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Abstract

We study exit decisions of duopolists from a stochastically declining market. Over time, firms privately learn about market conditions from observing the stochastic arrival of customers. Exit decisions are publicly observed; thus the model features both observational and private learning. We assume that a larger firm is more likely to have customers and hence has better information about market conditions than does a smaller rival. We provide sufficient conditions for either the smaller or the larger firm to be the first to exit the market in the unique equilibrium. Because of observational learning, exiting may be a firm's dominant action since continuing operation would bring too much of a good news to the rival, leading it to further postpone its exit. Uniqueness then follows from iterated conditional dominance.

Keywords: Duopoly, Exit, Private Learning, War of Attrition

JEL Codes: C73, D21, D43, D82, D83

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1 Introduction

The empirical evidence on exit from a declining industry has shown that the relationship between firm size and exit patterns varies across industries and may depend on industry-specific characteristics. Some studies have found higher rates of exit for small firms (see, for example, Lieberman, 1990). Other studies have documented that in mature stages of the industry life cycle, and particularly in technically advanced industries, smaller-scale firms are not necessarily confronted with a lower likelihood of survival relative to their larger counterparts (see Agarwal and Audretsch, 2001). As Besanko et al. (2010) point out, the existing theoretical literature fails to explain why "the swing producer," that is, the producer that adjusts to fluctuations in demand, "can be either the large firm or the small firm."

The theoretical literature has modeled strategic exit from a declining industry using the war of attrition paradigm, predicting that a stronger firm can force a weaker firm to exit first. Starting with the seminal contributions of Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986), and Fine and Li (1989) (see also Murto, 2004), the majority of papers have identified a firm's strength with its profit flow: a firm is stronger than its competitor if its profit flow is greater. To the best of our knowledge, no existing model explains why sometimes the firm that survives the industry decline is the one with the lowest profit flow.

Our paper offers a novel explanation for the seemingly unsettled correlation between profits and likelihood of survival. When a firm privately learns about the profitability of the industry, for example, from sales data, its strategy—whether it exits or not—conveys information to its competitor. As a result, a firm's relative strength is determined not only by its profit flow but also by the information externalities generated by its actions.

To model dynamic selection in a declining industry, we consider an irreversible timing game. Initially, each duopolist earns positive expected profits; the industry randomly transitions to a declining stage of its life cycle, unbeknown to either firms. In this declining stage, the duopoly loses viability as the expected profit of each duopolist becomes negative. Relying on the canonical exponential bandit framework, firms privately learn about the profitability of the industry by observing their customer arrivals. We focus on the case in which firms are asymmetric in that they have different customer arrival rates and hence learn at different speeds.[1] Since customer arrivals are privately observed and exit decisions are public, the model features both private and observational learning, akin to an incomplete information war of attrition.

First, we show that there always exists an equilibrium in which one of the two firms exits first with probability one. To determine which firm survives, we consider each firm's best reply to the other never exiting. In our model, the first exit time when playing this best reply[2] determines a firm's strength. The smaller the first exit time is, the weaker the firm. There always exists an equilibrium in which the weaker firm exits first with probability one acting solely on the base of its private information and never learns anything from the stronger firm,

Intuitively, the stronger firm has incentives to wait for the news revealed at the weaker firm's first exit time. In addition, if the weaker firm does not exit at that time, the stronger firm's incentive to remain in the market is reinforced. The weaker firm, by staying in the market, sends a signal that is against its own interest and that makes the stronger firm more optimistic about the market conditions—yet, in equilibrium, it cannot avoid doing so. As a result, the weaker firm is discouraged from remaining in the market longer compared to the case in which it expects never to enjoy monopoly profits nor to benefit from observing the other firm's action.

We show that there is a non-monotone relationship between the speed of private learning (or, equivalently, the firm's expected profit flow) and the firm's strength in the war of attrition (or, equivalently, its first exit time). The non-monotone relationship arises because of two countervailing forces. On the one hand, the higher the customer arrival rate is, the faster the firm becomes pessimistic about the market conditions if no customer shows up. On the other hand, the higher the customer arrival rate is, the higher the expected profit flow, and the stronger the incentive to remain in the market for any belief regarding the state of the industry.

Second, we provide sufficient conditions for the equilibrium in which the weaker firm exits first to be the unique equilibrium of the game. In light of the non-monotonicity between a firm's customer arrival rate and its strength, in the unique equilibrium either the larger or the smaller firm survives. As both predictions have re-

[1] We believe the mechanism at play is best illustrated in this set up because whenever firms are sufficiently asymmetric, there exists a unique equilibrium. However, the equilibria we construct are also equilibria of the symmetric game.

[2] This is the earliest time when the firm exits with positive probability along the path induced by the best reply.

ceived some empirical support, our model sheds light on how industry characteristics can affect the equilibrium outcome. Roughly, if there is a high degree of uncertainty, i.e., both firms have little information about the market conditions, then the smaller firm exits first with probability one; if the larger firm has (sufficiently) precise information, it exits first with probability one.

The proof relies on iterated deletion of (conditionally) dominated strategies à la Shimoji and Watson (1998). In principle, to identify dominated strategies, one needs to compute the beliefs of a firm for any given strategy of the rival. This computation proves to be difficult in our setup because higher-order beliefs play a key role, not only because a firm needs to forecast its opponent's action but also because firms' private signals are correlated. Standard techniques do not apply: since the underlying state of the world evolves over time, it is not possible to simplify the dynamic inference problem by decomposing the posterior belief into two single-dimensional statistics, i.e., a private belief and a public belief, such as in Foster and Viswanathan (1996) and Rosenberg, Solan, and Vieille (2007), as discussed in Section 4.2.

Our approach is radically different; we provide a recursive lower bound to a firm's posterior belief about the prevailing state in any equilibrium of the game. We believe that this approach can be applied to other models with private learning or private monitoring. In a first step, we compute a lower bound to the stronger firm's posterior belief conditional on the weaker firm not using a strictly dominated strategy. In a second step, we use this lower bound to identify an initial interval of time when continuing operations is a conditionally dominant strategy for the stronger firm, irrespective of its private history. In a third step, we show that exiting is initially dominant for the weaker firm whenever the time elapsed since last observing a customer is sufficiently long. We also show that, in the special case of conclusive news, the stronger firm's inference problem exhibits a recursive structure, which allows us to generalize the dominance argument to countably many rounds of deletion, concluding that the stronger firm never exits in equilibrium.

Inconclusive news disrupts the recursive properties of the inference problem. Our proof relies instead on a limit argument. In the limit as the weaker firm's information becomes arbitrarily precise, along any history, the stronger firm bases its inference only on observational learning, allowing us to conclude that there exists a unique equilibrium. Moreover, we show that introducing asymmetry in other dimensions such as discount rate, cost, or revenue does not change our main results.

Last, we show that our model is equivalent to an investment game in which competing firms privately learn over time about the comparative profitability of an innovation and decide when to invest in it. Our results can provide insights into the dynamics of investment in disruptive innovation à la Christensen (1997) or into the dynamics of pharmaceutical R&D (see, for example, Krieger, 2020).

1.1 Literature Review

Our work is closely related to the theoretical literature analyzing exit through the lens of the war-of-attrition paradigm. Ghemawat and Nalebuff (1985, 1990) study disinvestment in declining industries when the demand shrinks deterministically over time. Applying a backward induction argument, Ghemawat and Nalebuff (1985) show that in the unique equilibrium the larger firm exits first because it is unable to adjust capacity and loses viability more quickly. Murto (2004) shows that the main insights of Ghemawat and Nalebuff (1985) carry over to the case of stochastic market decline and a general payoff structure,[3] and there always exists an equilibrium in which the firm with the lowest expected profit flow exits first. In contrast to our model, in Murto (2004), signals about the underlying uncertainty, which is modeled as a geometric Brownian motion, are public, precluding signaling effects.

Incomplete information in a war of attrition has been studied in Fudenberg and Tirole (1986), who characterize the unique equilibrium of the exit game when firms have private information about their outside option or their cost. In contrast to Fudenberg and Tirole (1986), in our model information is interdependent; thus, higher-order beliefs are relevant not only to form beliefs about the strategy of the opponent but also to assess the prevailing state of the world. Takahashi (2015) empirically estimates the model of Fudenberg and Tirole (1986) using the US movie theater industry to quantify the welfare loss from strategic delay. In the same spirit, our paper provides a tractable framework to estimate the welfare implications of observational learning in the presence of interdependent private information.

Within the broader literature on stopping games, our paper is closely related to Rosenberg, Solan, and Vieille (2007) and Murto and Välimäki (2011). While these papers are concerned with different questions, they feature, similar to ours, observational learning and irreversible action, but they do not incorporate payoff

[3] See also the discrete-time model of Fine and Li (1989).

externalities. In contrast, Hopenhayn and Squintani (2011) and Gorno and Iachan (2020) feature independent private information and payoff externalities.[4]

More recently, Awaya and Krishna (2021) show that information can be a strategic disadvantage in an R&D race with private information.[5] They show that under some parametric restrictions, in the unique equilibrium, the better-informed firm exits the race more frequently and has lower payoffs.[6] According to the authors, the less-informed firm's incentive to wait and learn from the better-informed firm is the main driver of their results.

In contrast to Awaya and Krishna (2021), in our model, firm's are not only asymmetric in their information, but also in their payoffs. As in canonical models of market experimentation, customers bring information and revenues; the assumption is aimed to capture a positive correlation between a firm's quality of information and its profit flow. In light of Awaya and Krishna's result, one may expect information to be sometimes a strategic disadvantage. The question is, however, whether this effect can overturn what the existing literature (e.g., Ghemawat and Nalebuff, 1985 and Murto, 2004) predicts, that is, that the firm with the lowest profit flow exits first. We address this question by identifying the appropriate notion of strength and show that the relationship between strength and the speed of private learning is non-monotone, specifically, it is single-peaked. At same time, as compared to Awaya and Krishna (2021), our result clarifies the role of the signaling consequences of a firm's action in determining the benefits and drawbacks of being the better- or less-informed firm.

Our model is also related to the literature on strategic experimentation with exponential bandits. As in Keller and Rady (2010), firms learn via inconclusive good news, but as in Keller and Rady (1999, 2003), the underlying state of the world

[4] While these papers focus on symmetric setups and equilibria, in our equilibrium, as in the asymmetric equilibria of the model of Kirpalani and Madsen (2020), only one firm reaps the benefit of observational learning.

[5] Relatedly, Moscarini and Squintani (2010) investigate the role of private information in a winner-take-all R&D race in a model which, barring the technical details of the information structure, is identical to the one of Awaya and Krishna (2021). Moscarini and Squintani (2010) show that the equilibrium features waves of exits and a "survivors curse," that is, on the equilibrium path, a firm may regret not having exited earlier.

[6] Similarly, Chen and Ishida (2020) study an asymmetric war of attrition with independent types and dynamic private learning and show that in some equilibrium the less-efficient firm wins more often. See also Kim and Lee (2014) who study the affect of information acquisition in war of attrition.

changes over time.[7] Rosenberg et al. (2013) and Heidhues et al. (2015), who analyze experimentation models with private learning, are also related.

1.2 Structure of the Paper

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 analyzes two public learning benchmarks: the case of observable decline and the case of publicly observable customers. Section 4 is devoted to the main results. Section 5 shows that the model is equivalent to an irreversible investment game with a first mover advantage, and Section 6 concludes.

2 Model

Time is continuous, and the horizon is infinite, $t \in [0, \infty)$. Two firms decide when to irreversibly exit a declining industry. Each firm's present discounted payoff from exiting the industry is normalized to 0.

The industry profitability is determined by a state of the world ω_t that can be either good or bad, $\omega_t \in \{G, B\}$. Initially, both firms attach probability one to the industry being profitable, $\omega_0 = G$. The industry irreversibly becomes unprofitable, unbeknown to the two firms, at some random time which is exponentially distributed with parameter $\gamma > 0$.[8]

Each active firm serves a stream of randomly arriving customers. In a duopoly, that is, as long as both firms are active, the customers of firm i arrive according to an inhomogeneous Poisson process with intensity $\lambda^i_{\omega_t}$, where $\lambda^i_G > \lambda^i_B \geq 0$. In a monopoly, that is, after firm j exits, the customers of firm i arrive at a rate $\lambda^1_{\omega_t} + \lambda^2_{\omega_t}$. Each firm privately observes its customer arrival while exit decisions are public.

[7] See also Khromenkova (2018).

[8] Our model generalizes to the case of $\gamma = 0$ and interior prior about the state of the world ω_0. The assumption that the distribution of the time when the state transitions is exponential is convenient but not essential. The results can easily be generalized, for example, to any distribution with a bounded hazard rate.

While active, each firm bears a flow cost c, and each customer yields a lump-sum revenue R. Firms discount the future at a common rate $r > 0$. We impose the following parametric assumptions.[9]

Assumption 1. *For $i = 1, 2$,*

$$\lambda_B^i R - c < 0.$$

Assumption 1 states that a duopolist's flow payoff is negative whenever $\omega_t = B$: it ensures that it can ever be optimal to exit. Furthermore, we assume that $\lambda_B^2 = \lambda_B^1 = \lambda_B$.

After one of the firms has exited, the remaining firm enjoys monopoly profits until it finds it optimal to exit as well, if ever. In fact, our model can accommodate both the case in which a monopolist's profit is always profitable and the case in which a monopolist's profit is negative whenever $\omega_t = B$, that is, $2\lambda_B R - c \lesseqgtr 0$.

A strategy for firm i dictates when to exit along any nonterminal history as a function of all the information available. Formally, a strategy of firm i, σ^i, is a stopping time adapted to the filtration generated by the inhomogeneous Poisson process of customer arrivals, N_t^i, and the exit decision of player j, i.e., whether and when player j has exited.

Given a strategy profile (σ^1, σ^2), the payoff of firm i can be written as:

$$\mathbf{E}^{(\sigma^1,\sigma^2)}\left[\int_0^{\sigma^i} e^{-rt}\left(\lambda_{\omega_t}^i R + \mathbf{1}_{\{\sigma^j < t\}}\lambda_{\omega_t}^j R - c\right) \mathrm{d}t\right]. \tag{1}$$

We focus on Perfect Bayesian equilibria of the stochastic timing game. However, we do not explicitly specify beliefs and behavior off the equilibrium path because they play no role in sustaining on-path behavior, given that the payoff from exiting is independent of the behavior of the remaining firm and exit is irreversible.[10]

As in canonical bandit models (e.g., Rothschild, 1974), customers bring revenue and information. Conceptually, our results rely on the positive correlation between the expected profit of a firm and the precision of its information. For example, one

[9] We could allow the flow cost to depend on the presence of a competitor, as long as the expected profit conditional on the state is strictly higher in a monopoly than that in a duopoly. For a discussion on asymmetries, see Section 4.4.

[10] In our game, any Nash equilibrium is outcome-equivalent to a Perfect Bayesian equilibrium.

could argue that a larger firm enjoys higher profits because of higher markups and economies of scale[11] and that it has more precise information as a result of a more sophisticated market research department.

3 Benchmarks

To put our results into perspective, we start by discussing two benchmarks. In the first benchmark, the state transition is observable. In the second, firms do not observe the state transition, but they observe each others' customers.

3.1 Observable Decline

As long as the state is good, i.e., $\omega_t = G$, it is dominant for both firms to remain in the market. If $2\lambda_B R - c > 0$,[12] the continuation game after the state transitions is a standard war of attrition, as in Hendricks et al. (1988), which is known to have a multiplicity of equilibria.

Specifically, it has two pure strategy asymmetric equilibria. In each of them, one of the firms exits as soon as the state transitions while the other firm never exits. There exists a mixed strategy symmetric equilibrium in which both firms exit at the constant rate

$$-r\frac{\lambda_B R - c}{2\lambda_B R - c}$$

such that the cost of waiting, $c - \lambda_B R$, equals the benefit, $\varsigma\left(2\lambda_B R - c\right)/r$, where ς is the equilibrium exit rate of each firm.

Hence, a model with observable state transition is silent about how industry characteristics affect the likelihood that the smaller or the larger firm survives the industry decline. In contrast, with private learning, the equilibrium is unique, and we can identify the conditions under which either the smaller or the larger firm exits first.

[11] The empirical literature finds higher markups for larger firms, e.g., De Loecker and Warzynski (2012), Edmond et al. (2018), Autor et al. (2020), and Boar and Midrigan (2020).

[12] Trivially, if $2\lambda_B R - c \leq 0$, in equilibrium firms exit as soon as the state transitions.

3.2 Public Learning

If firms observe each other's customers, the model is closely related to the model of Murto (2004). Despite the difference in the stochastic process governing the underlying state of the world, the main insights of Murto (2004) carry over to our setup, as formalized by the following proposition. For simplicity, we focus on the case of conclusive news, $\lambda_B = 0$.

Proposition 1. *If both firms observe each other's customers and $\lambda_G^2 > \lambda_G^1 > c/R$,*[13] *there always exists an equilibrium in which firm 2 never exits first. If λ_G^1 is low enough, this is the unique equilibrium provided that R is high enough (or c is low enough) and $r > \lambda_G^1 + \lambda_G^2$.*

As in Murto (2004), whenever the game has a unique equilibrium, the larger firm forces the weaker one to exit first.[14] In other words, a firm's "strength" is monotone in its profit flow: the larger λ_G^i is, the longer the firm is willing to remain in the market. When instead there exist multiple equilibria and, in particular, there exists an equilibrium in which the smaller firm never exits first, there also exists a mixed strategy equilibrium.

As we discuss in the Appendix, in the mixed strategy equilibrium, the firm with the higher customer arrival rate exits with positive probability at a certain belief; for lower beliefs, both firms exit at a positive rate.[15] In equilibrium, the rate at which a firm exits makes the opponent indifferent between exiting and remaining in the market. Consequently, as in a nondegenerate equilibrium of a complete information war of attrition, the firm with the larger customer arrival rate has a lower probability of survival.

It has been argued that it is odd that in the mixed strategy equilibrium, the firm with the smaller customer arrival rate (i.e., the largest cost of fighting) wins more frequently. Hence, when multiple equilibria exist, the most realistic equilibrium seems to be the one in which the firm with the larger customer arrival rate never exits first. (See, for example, Kornhauser et al., 1989.) In a sense, our model provides a rationale

[13] We discuss the bound $\lambda_G^1 > c/R$ in Lemma 1 below.

[14] Notice, however, that Murto (2004) shows uniqueness within the class of Markov equilibria while we provided suffiicient conditions for uniqueness of a Nash equilibrium both in the case of public learning and in the case of private learning. (See Theorem 2.)

[15] The existence of a mixed-strategy equilibrium, seemingly in contrast to Georgiadis et al. (2020), depends on the stochastic properties of the belief process; in Georgiadis et al. (2020) the belief is a Brownian motion.

for the larger firm to concede more often in equilibrium: interdependent values and observational learning.

4 The Private Learning Game

This section discusses our main results. We first introduce a notion of "strength" in our war of attrition game by means of analyzing a specific best-reply problem. Second, we construct two candidate equilibria. Third, we provide sufficient conditions for each of them to be the unique strategy profile that survives iterated deletion of (conditionally) dominated strategies, implying equilibrium uniqueness. Last, we discuss the main forces at play.

4.1 A Best-Reply Problem

Suppose firm j adopts the strategy of never exiting the market, and consider the best-reply problem of firm $i \neq j$. The problem of firm i can be written as a standard optimal stopping problem:

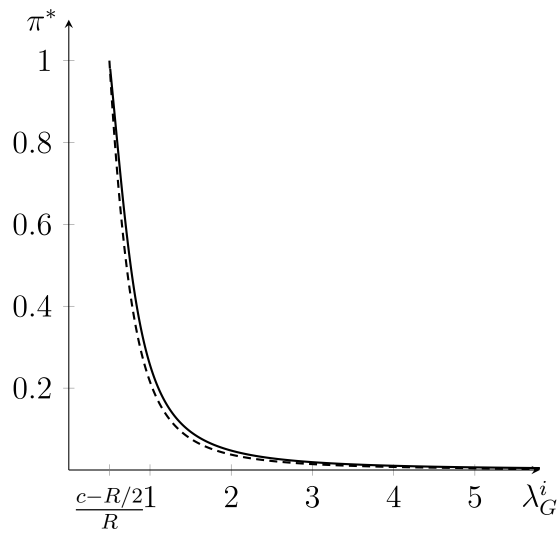
$$\sup_{\tau} \mathbf{E}\left[\int_0^{\tau} e^{-rt}\left(\lambda^i_{\omega_t} R - c\right) \mathrm{d}t\right].$$

The problem of firm i is Markov in its posterior belief about the prevailing state and the best response takes a simple form: it prescribes exiting as soon as the posterior belief falls below some cutoff $\pi^*(\lambda^i_G)$. Define $\tau^*(\lambda^i_G)$ as the earliest time firm i exits with positive probability along the path induced by the best-reply strategy, that is,

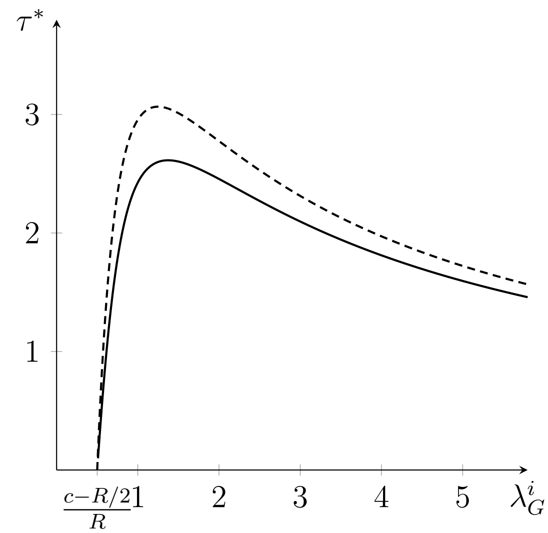
$$\Pr\left[\omega_{\tau^*(\lambda^i_G)} = G \mid N^i_{\tau^*(\lambda^i_G)} = 0\right] = \pi^*(\lambda^i_G).$$

In other words, along the history with no customers, firm i exits at time $\tau^*(\lambda^i_G)$. (Recall that N^i_t denotes the inhomogeneous Poisson process of customer arrivals of firm i.)

In the special case of conclusive news, i.e., $\lambda_B = 0$, $\tau^*(\lambda^i_G)$ fully characterizes the best reply of firm i. Because the posterior belief about the prevailing state jumps to one whenever the firm observes a customer, the best reply prescribes exiting as soon as no customers have been observed for an uninterrupted amount of time of length $\tau^*(\lambda^i_G)$.



(a) Cutoff belief



(b) First exit time

Figure 1: Best reply for $(c, R, r, \gamma) = (1/2, 1, 1/10, 1/5)$. The solid line indicates the case of conclusive news, $\lambda_B = 0$. The dashed line indicates a case of inconclusive news, $\lambda_B = 1/5$.

The following lemma characterizes how $\pi^*(\lambda_G^i)$ and $\tau^*(\lambda_G^i)$ change with λ_G^i.

Lemma 1.

(i) $\pi^*(\lambda_G^i) < 1$ *if and only if* $\lambda_G^i R > c$.

(ii) $\tau^*(\lambda_G^i)$ *is single-peaked and* $\lim_{\lambda_G^i \to \infty} \tau^*(\lambda_G^i) = 0$.

Intuitively, if λ_G^i is sufficiently low, it is too unlikely that the firm will be able to serve enough customers to make it worthwhile to remain in business. In light of (i), in the remainder of the paper, we assume that $\lambda_G^i R > c$, $i = 1, 2$.

The non-monotonicity, illustrated in the right panel of Figure 1, is due to two countervailing forces. On the one hand, the higher λ_G^i is, the higher the marginal benefit from remaining in the market at any given belief. In fact, the cutoff belief $\pi^*(\lambda_G^i)$ is decreasing in λ_G^i, as in the left panel of Figure 1. On the other hand, the higher λ_G^i is, the faster the firm becomes pessimistic about the market conditions.[16] This observation is at the core of our main results.

[16] The result is reminiscent of Halac, Kartik, and Liu (2016) and Bobtcheff and Levy (2017). Conceptually, our result generalizes theirs to a setup with inconclusive news and changing state.

In contrast to the case of privately observed customers, when learning is public, as in Section 3.2, a firm's first exit time is monotone in the rate of arrival of customers, that is, it is monotone in its profit flow. In fact, the single-peakedness in Lemma 1 relies on the fact that as λ_G^i increases, both the profit flow and the speed of learning increase.[17] With observable customers, the second force plays no role and there always exists an equilibrium in which the firm with the higher customer flow never exits first. This non-monotonicity is the main reason why our model delivers different predictions from those of existing models.

To simplify the exposition of our results, we say that firm i is *stronger* than firm $j \neq i$ if the first exit time of firm i is larger than the first exit time of firm j, that is, $\tau^*(\lambda_G^i) > \tau^*(\lambda_G^j)$. In the next section, we show that there always exists an equilibrium in which the stronger firm survives the industry decline, that is, the weaker firm exits first with probability one.

It is worth contrasting our definition of strength with Murto's (2004), which is in line with the standard war of attrition's idea that the stronger player has a lower fighting cost. Murto's (2004) definition of strength depends on a firm's behavior once it becomes a monopolist and implies that a firm is stronger if it has a higher profit flow. In contrast, as it will become clear in the next section, our definition of strength captures the signaling consequence of a firm's action; because of the non-monotonicity stated in Lemma 1, the firm with the higher profit flow may not be the stronger one.

4.2 A pure strategy equilibrium

We now show the existence of an equilibrium in which the weaker firm exits first with probability one. The section proceeds through a sequence of observations and the main result is formalized in Theorem 1 below.

Since $\pi^*(\lambda_G^i)$ is the optimal exit cutoff under the most pessimistic scenario in which the other firm never exits first, in equilibrium, exiting when the posterior belief is larger than $\pi^*(\lambda_G^i)$ is a dominated strategy, as formalized below.

[17] It can also be shown that if one normalizes the lump-sum payoff so that the expected flow is independent from the rate of arrival of news, the first exit time monotonically decreases in the learning speed. Details are available upon request.

Lemma 2. *In any equilibrium, if firm i exits with positive probability at time t along some history in which firm j is still active, then*

$$\Pr[\omega_t = G \mid (N^i_s)_{s\le t}, \sigma^j \ge t] \le \pi^*(\lambda^i_G).$$

Proof. Consider firm i at time t following the private history $(N^i_s)_{s\le t}$, and assume

$$\Pr[\omega_t = G \mid (N^i_s)_{s\le t}, \sigma^j \ge t] > \pi^*(\lambda^i_G).$$

Hence, if the firm remains in the market until $t + \mathrm{d}t$, the expected payoff it collects in $[t, t + \mathrm{d}t)$ is bounded below by

$$\Bigg(- c + \big(\Pr[\omega_t = B \mid (N^i_s)_{s\le t}, \sigma^j \ge t]\lambda_B + \Pr[\omega_t = G \mid (N^i_s)_{s\le t}, \sigma^j \ge t]\lambda^i_G\big) \cdot \left(R + v\left(\frac{\pi^*(\lambda^i_G)\lambda^i_G}{\pi^*(\lambda^i_G)\lambda^i_G + (1-\pi^*(\lambda^i_G))\lambda_B}\right)\right)\Bigg)\,\mathrm{d}t > 0,$$

where $v : [0, 1] \to \mathbf{R}$ denotes the value function associated with the best-reply problem in Section 4.1. The bound follows from a few observations. First, the last term on the left-hand side is a lower bound to the expected continuation payoff after observing a customer: firm i can only benefit from firm j using a strategy other than never exiting. Second, in $[t, t + \mathrm{d}t)$, firm j may exit the market, but because the expected payoff in the continuation game is weakly positive, we can omit the corresponding term. Last, by definition of $\pi^*(\lambda^i_G)$, the inequality holds for any $\Pr[\omega_t = G \mid (N^i_s)_{s\le t}, \sigma^j \ge t] > \pi^*(\lambda^i_G)$. Therefore, the result follows. □

In any equilibrium, a firm's belief about the underlying state of the world evolves because of private and observational learning. In light of existing results, such as those of Rosenberg et al. (2007), one may expect to be able to decompose a player's posterior belief into two single-dimensional statistics, i.e., a private belief and a public belief. Unfortunately, there exist no single-dimensional statistics that, combined with the private belief, yield the posterior belief about the prevailing state of the industry.

Intuitively, because the state is not perfectly persistent, the fundamental uncertainty concerns not only whether the industry has already become unprofitable but also when that happened. In fact, conditional on the prevailing state being bad, i.e.,

$\omega_t = B$, players' private signals are correlated, making the standard decomposition technique inapplicable.

As a result, the relevant information for firm i cannot be summarized by its private belief about the prevailing state and the status of the other firm (active or not). The second-order belief of firm i, that is, its distribution over the private belief firm j, affects firm's i posterior about the current state of the world in a more complicated manner than in Rosenberg et al. (2007).[18]

Our first main result identifies an equilibrium in which firms' inference problems are uncomplicated. A class of strategies that generate a simple inference problem are cutoff strategies. According to a cutoff strategy, for some measurable function $p_t : [0, \infty) \to [0, 1]$, a firm exits with probability one at the first-passage time of its posterior belief under p_t. For any $p > 0$, let σ_p^i be the pure strategy according to which firm i adopts a time-independent cutoff p. Let σ_0^i be the strategy that prescribes never exiting.

Theorem 1. *Fix the customer arrival rate of firm 1. If the customer arrival rate of firm 2 is high enough, there exists an equilibrium in which firm 2 (the larger firm) exits first with probability one, i.e.,* $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ *is an equilibrium.*

Proof. Here, we provide the proof for the case of conclusive news, $\lambda_B = 0$. The proof for the case of inconclusive news is relegated to the Appendix. We claim that if $\tau^*(\lambda_G^1) > \tau^*(\lambda_G^2)$, then $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium. The result then follows from Lemma 1. First, by definition, $\sigma^2_{\pi^*(\lambda_G^2)}$ is a best reply to σ_0^1. Second, by the proof of Lemma 2, if for some $t \geq 0$, $\Pr[\omega_t = G \mid (N_s^1)_{s \leq t}, \sigma^2 \geq t] > \pi^*(\lambda_G^1)$, it is dominant for firm 1 to remain in the market at t. Even if firm 1 did not observe any customer in $[0, \tau^*(\lambda_G^2))$, its belief $\Pr[\omega_t = G \mid (N_s^1)_{s \leq t}, \sigma^2 \geq t]$ would jump upward at $\tau^*(\lambda_G^2)$ and remain at some value strictly higher than $\pi^*(\lambda_G^1)$ as long as it does not observe any arrival and firm 2 does not exit. In fact, along a history with no customers, as long as firm 2 does not exit, the belief of firm 1 is constant whenever the last customer was observed more than $\tau^*(\lambda_G^1)$ amount of time ago.[19] It follows that σ_0^1 is a best reply to $\sigma^2_{\pi^*(\lambda_G^2)}$. □

[18] Cisternas and Kolb (2020) show that in a private monitoring setup, the second-order beliefs can be decomposed in a similar vein. Yet, in their setup the first- and second-order beliefs are determined by a finite-dimensional sufficient statistic.

[19] We discuss this fact at length when explaining Figure 2 below.

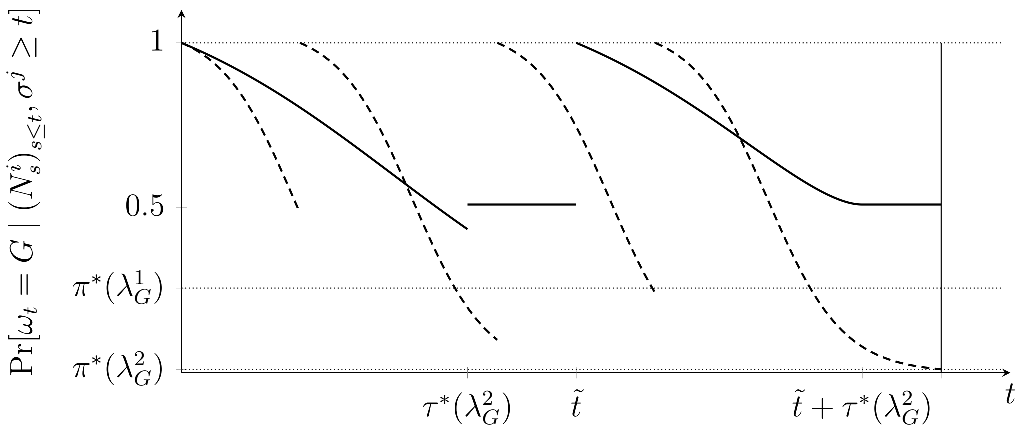
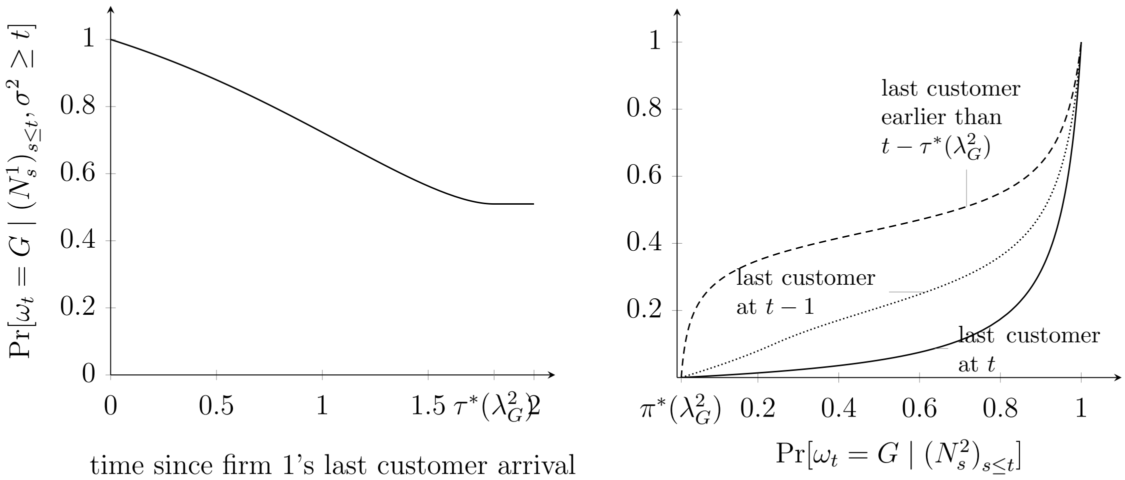


Figure 2: Example of equilibrium belief trajectories for $(c, R, r, \gamma, , \lambda_G^1, \lambda_G^2, \lambda_B) = (1/2, 1, 1/10, 1/5, 4, 1, 0)$. The solid line is the belief trajectory of firm 1. The dashed line is the belief trajectory of firm 2. The vertical line demarcates the time at which firm 2 exits.

In words, if the arrival rate of firm 2 is high enough, firm 1 is the stronger one, and survives the industry decline. The theorem provides sufficient conditions for the existence of an equilibrium in which firm 2 exits first with probability one. However, the proof above shows something stronger: in the case of conclusive news, for any pair $(\lambda_G^1, \lambda_G^2)$ there always exists an equilibrium in which the weaker firm exits first with probability one. That is, either $(\sigma_0^1, \sigma_{\pi^*(\lambda_G^2)}^2)$ or $(\sigma_{\pi^*(\lambda_G^1)}^1, \sigma_0^2)$ is an equilibrium, depending on whether $\tau^*(\lambda_G^1) \gtrless \tau^*(\lambda_G^2)$. Hence, it also guarantees the existence of equilibrium in the case of conclusive news.

To gain some insight into the learning dynamics, Figure 2 illustrates a possible realization of belief paths for the equilibrium in Theorem 1 under the assumption that $\lambda_G^2 > \lambda_G^1 > \lambda_B = 0$ and firm 1 is the stronger firm, that is, $\tau^*(\lambda_G^1) > \tau^*(\lambda_G^2)$. First, notice that in equilibrium, firm 2 never benefits from observational learning: the evolution of firm 2's belief at any point in time is uniquely driven by private learning. Second, in the interval $[0, \tau^*(\lambda_G^2))$, observational learning plays no role for firm 1 either. In fact, no firm is supposed to exit, and both base their assessment of the market profitability on their private signals only.

In the equilibrium outcome illustrated in Figure 2, firm 2 does not exit at $\tau^*(\lambda_G^2)$ because it observes a customer before that time. At $\tau^*(\lambda_G^2)$, firm 1's belief jumps upward as firm 2 not exiting reveals that it has observed at least a customer in $[0, \tau^*(\lambda_G^2))$. In the example, firm 1 does not observe any customer in $[0, \tilde{t})$. As a



(a) First-order belief

(b) Second-order belief

Figure 3: The first- and second- order belief of firm 1 in the equilibrium $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ for $(c, R, r, \gamma, \lambda_G^1, \lambda_G^2, \lambda_B) = (1/2, 1, 1/10, 1/5, 4, 1, 0)$. On the left, the posterior about the prevailing state at some time $t > \tau^*(\lambda_G^2)$ as a function of the time elapsed since last observing a customer. On the right, firm 1's distribution over firm 2's posterior belief at some $t > \tau^*(\lambda_G^2)$.

result, at any $t \in [\tau^*(\lambda_G^2), \tilde{t})$, the belief of firm 1 about the prevailing state of the world, as well as its second-order belief, are constant.[20] (See also Figure 3.) Firm 2 not exiting at some $t \in [\tau^*(\lambda_G^2), \tilde{t})$ reveals that firm 2 has observed its last customer no earlier than $t - \tau^*(\lambda_G^2)$; otherwise its belief would have fallen below $\pi^*(\lambda_G^2)$ at some time before t. Hence, firm 2 not exiting at $t \in [\tau^*(\lambda_G^2), \tilde{t})$ also reveals that $\omega_{t-\tau^*(\lambda_G^2)} = G$. Consequently, from the point of view of firm 1, the lack of customers in $[0, t - \tau^*(\lambda_G^2))$ becomes irrelevant as far as its belief about the prevailing state is concerned. Intuitively, firm 1 knows that firm 2 has "fresher" news and can discard part of the information contained in its private history. This explains why the first-order belief in the left-panel in Figure 3 eventually plateaus.

Moreover, in the outcome shown in Figure 2, as soon as firm 2 exits, firm 1 follows suit. As in Moscarini and Squintani (2010), the equilibrium may display exit waves because information is reveled in a burst when a firm exits. In contrast, information is revealed gradually at any point in time after the first exit time of the weaker firm, as long as both firms remain in the market.

[20]The first- and second-order beliefs of firm 1 would not be constant in the case of inconclusive news, $\lambda_B \neq 0$.

4.3 Equilibrium Uniqueness

In general, the equilibrium identified in Section 4.2 is not the unique one. If firms' customer arrival rates are sufficiently similar, there also exists an equilibrium in which the firm with the larger first exit time exits first with probability one. For example, for the parameters in Figure 2, both $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ and $(\sigma^1_{\pi^*(\lambda_G^1)}, \sigma_0^2)$ are equilibria of the game. In fact, if firm 1 exits as soon as its private belief falls below the benchmark cutoff belief $\pi^*(\lambda_G^1)$, i.e., plays the strategy $\sigma^1_{\pi^*(\lambda_G^1)}$, firm 2 has incentives to remain in business at any point in time and along any history before firm 1's exit because its continuation payoff is strictly positive. In other words, firm 2, anticipating that firm 1 will eventually exit, is willing to remain in the market at beliefs below the cutoff $\pi^*(\lambda_2^G)$.[21]

When payoff externalities are absent, in light of Rosenberg et al. (2007) , it is natural to expect all the equilibria to be in cutoff strategies. The non-monotonicity of the continuation payoff (see footnote 21) as well as the results by Murto (2004) suggest that this is not true in our setup. It is easy to construct simple mixed strategy equilibria. For example, for some parameters, the following strategy profile is an equilibrium. Firm 2 exits with positive probability at time $\tau^*(\lambda_G^2)$ whenever $N^2_{\tau^*(\lambda_G^2)} = 0$. If it does not exit at that time, it never exits ever after. Firm 1 adopts the strategy $\sigma^1_{\pi^*(\lambda_G^1)}$. The probability with which firm 2 exits is chosen so that firm 1's best-reply to it makes firm 2 indifferent between exiting at $\tau^*(\lambda_G^2)$ and never exiting ever after along the history with no customers.

Nevertheless, we are able to show that under appropriate conditions the game has a unique equilibrium, specifically, there exists a unique strategy profile that survives iterated deletion of conditionally dominated strategies.

Theorem 2.A. *Assume that $2\lambda_B R - c \leq 0$. In the case of both conclusive and inconclusive news, i.e., $\lambda_B \geq 0$, for any λ_G^1, there exists a $\overline{\lambda}_G^2 > \lambda_G^1$ such that for $(\lambda_G^1, \lambda_G^2)$, $\lambda_G^2 > \overline{\lambda}_G^2$, $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of conditionally dominated strategies; hence, in the unique equilibrium, the larger firm exits first (with probability one).*

[21]Interestingly, however, on the path induced by the equilibrium $(\sigma^1_{\pi^*(\lambda_G^1)}, \sigma_0^2)$, before $\tau^*(\lambda_G^1)$, the continuation payoff of firm 2 is sometimes non-monotone in its posterior belief. In fact, as illustrated in Figure 3, a firm's customer arrivals affect its second-order belief, and hence the expected exit time of the rival.

Theorem 2.B. *In the case of conclusive news, i.e.,* $\lambda_B = 0$*, there exists an open set of pairs* $(\lambda_G^1, \lambda_G^2)$*,* $\lambda_G^2 < \lambda_G^1$*, under which* $(\sigma_0^1, \sigma_{\pi^*(\lambda_G^2)}^2)$ *is the unique strategy profile that survives iterated deletion of conditionally dominated strategies, provided that* R *is high enough (or* c *is low enough) and that* r *and* γ *are high enough; hence, in the unique equilibrium, the smaller firm exits first (with probability one).*

Intuitively, the lower $\tau^*(\lambda_G^2)$ is, the more influential firm 2's action, that is, the stronger the inference drawn by firm 1 by observing firm 2 not exiting the market. In other words, the lower $\tau^*(\lambda_G^2)$ is, the more firm 1 benefits from observational learning. Because observing firm 2 not exiting brings good news, this strengthens incentives for firm 1 to remain in the market. As a consequence, if this "signaling disadvantage" is large enough, in the unique equilibrium, firm 1 eventually becomes the monopolist.

To put it differently, private learning generates a discouragement effect. Firm 2 would be willing to stay in the market at a belief lower than the single-player cutoff only if it expected to eventually become a monopolist. However, by continuing operations firm 2 makes the opponent more optimistic and delays its exit. As a result, anticipating a longer duopoly phase, it is discouraged from remaining in the market.

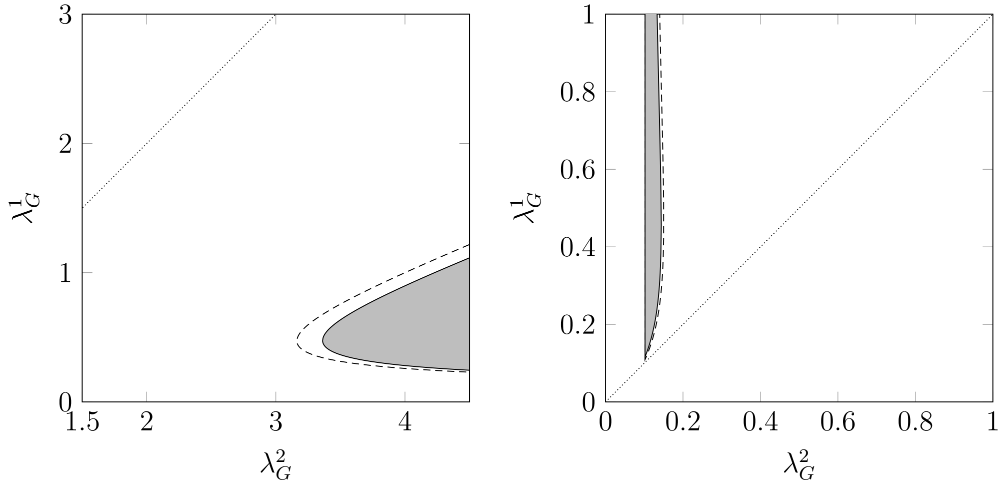
Theorem 2 provides sufficient conditions for firm 2 to be the first to exit in the unique outcome that survives iterated deletion of conditionally dominated strategies.[22] Interestingly, depending on the parameters the larger or the smaller firm eventually becomes the monopolist.

As shown in the Appendix, our result does not rely on the assumption that a sufficiently pessimistic monopolist finds it optimal to exit, in that we can show that even when being monopolist is always profitable, i.e., $2\lambda_B R - c > 0$, there exists a non-empty set of parameters such that the weaker firm exits first in the unique equilibrium of the game.[23]

Theorem 2 can be summarized as follows: under some parametric restrictions, in the unique equilibrium the stronger firm, as defined in Section 4.1, survives. The novel observation is the non-monotonic relationship between strength and profit flow, which may be seen as a proxy for firm size.

[22]While Theorem 2 gives sharp predictions, the sufficient parametric conditions in part B are somewhat not tight. We believe that the uniqueness results extend to a larger set of parameters but we were unable to derive tighter bounds.

[23]In fact, the statement is only slightly weakened. If $r > \gamma + \lambda_B$, we can still identify a set of pairs $\mathcal{L} \in (c/R, \infty) \times (c/R, \infty)$, $\lambda_G^2 > \lambda_G^1$, for which $(\sigma_0^1, \sigma_{\pi^*(\lambda_G^2)}^2)$ is the unique equilibrium, but in contrast to Theorem 2.A, $\text{proj}_1 \mathcal{L} \neq (c/R, \infty)$.



(a) Theorem 2.A (b) Theorem 2.B

Figure 4: In gray, the sets of pairs $(\lambda_1^G, \lambda_G^2)$ for which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda^1_G})$ is the unique equilibrium for $(c, R, r, \gamma, \lambda_B) = (1/10, 1, 7, 1/5, 0)$. The dashed line identifies the sets of pairs $(\lambda_1^G, \lambda_G^2)$ for which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda^1_G})$ is the unique equilibrium when $r = 20$.

Our results provide a rationale for why the technological features of an industry affect the relationship between firm size and the likelihood of survival, as shown by Agarwal and Audretsch (2001) (see also Lieberman, 1990). Agarwal and Audretsch (2001) argue and empirically show that what Geroski (1995) identifies as a stylized fact—that the likelihood of survival is greater for large firms than for small firms—does not hold in the mature phase of the industry life cycle and in high-technology industries.[24]

Figure 4 provides an illustration of the theorem; it identifies the set of pairs $(\lambda_G^1, \lambda_G^2)$ for which firm 2 exits first in the unique strategy profile that survives iterated deletion of dominated strategy. Our model predicts that if there is a high degree of uncertainty, that is, both firms have little information about the market conditions, the small firm exits first. (See right panel in Figure 4.) Intuitively, if firms' private information is of little use in predicting future profits, in line with Geroski (1995),

[24] Agarwal and Audretsch (2001) explain how the theory of strategic niches can explain this finding. We believe that our model can provide a more compelling justification for the finding, in light of the fact that confidential business information has been recognized to play a key role in market competition (see, for example, the FTC order designed to remedy the anticompetitive effects resulting from Broadcom Limited's acquisition of Brocade Communications Systems, https://www.ftc.gov/enforcement/cases-proceedings/171-0027/broadcom-limitedbrocade-communications-systems).

Theorem 2.B predicts that the firm with the lowest profit flow exits first in the unique equilibrium of the game. However, when the large firm has (sufficiently) precise information about the market conditions, it is the small firm that survives the industry decline, as in Agarwal and Audretsch (2001).

As the figure suggests, the set identified in Theorem 2.A is unbounded while the set identified in Theorem 2.B is bounded. That is, whenever the equilibrium is unique, generically, the larger firm exits first. Similarly, there is an asymmetry in the statements of the two parts of the theorem. We discuss these asymmetries below when we discuss the proof.

4.3.1 Discussion of the Proof

The proof relies on iterated deletion of (conditionally) dominated strategies.[25] The argument is somewhat involved; here, we illustrate the main ideas. First, we derive a (uniform) lower bound on the belief of firm 1 along any path induced by a strategy profile that survives iterated deletion of conditionally strictly dominated strategies in the case of conclusive news. Next, we use a limit argument to bound the belief in the case of inconclusive news.

The argument to derive the lower bound on firm 1's belief is divided in several steps. First, recall that remaining in the market is a dominant action for firm i whenever it attaches a probability higher than the cutoff $\pi^*(\lambda_G^i)$ to the prevailing state being good. (See Lemma 2.) We show that, under the assumptions of the theorem, for any strategy of firm 2 that prescribes exiting at some belief below $\pi^*(\lambda_G^2)$, firm 2 continuing operation always brings good news.[26] As a result, the belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$, making exiting before $\tau^*(\lambda_G^1)$ a dominated action.

Second, we argue that if the gap between the two first exit times is large enough, specifically, if $2\tau^*(\lambda_G^2) \ll \tau^*(\lambda_G^1)$, as in Figure 5, at any $t \in [0, 2\tau^*(\lambda_G^2))$, exiting is a conditionally dominant strategy for firm 2 whenever its belief falls below the cutoff $\pi^*(\lambda_G^2)$. Intuitively, firm 2 would be willing to remain in the market at a lower belief

[25] As explained by Shimoji and Watson (1998), the standard notion of dominance has little bite in extensive form games. In the Appendix, we explain how their definition of conditional dominance can be extended to our setup.

[26] When firm 2 uses a cutoff strategy, this is immediate. However, in contrast to Rosenberg et al. (2007), it is not necessarily true that any rationalizable strategy is a cutoff strategy (see Section 4.3). Hence, arguing that observing that the rival is still active always brings good news requires a more delicate argument.

only if it expected firm 1 to exit with positive probability at some future point in time. However, rationality implies that firm 1 does not exit until relatively late in the game, that is, until $\tau^*(\lambda_G^1)$, and hence firm 2 does not find it worthwhile to bear the expected losses to wait until then.

Third, at time $2\tau^*(\lambda_G^2)$, if firm 2 does not exit, firm 1 can infer that the belief of firm 2 never fell below the cutoff $\pi^*(\lambda_G^2)$ in $[0, 2\tau^*(\lambda_G^1))$. In the case of conclusive news, it is easy to identify the pair of firms' private histories that are consistent with them playing conditionally undominated strategies and with the public history of no exit and which, when combined, would give rise to the lowest posterior belief about the state at time $2\tau^*(\lambda_G^2)$. For firm 1, the "worst" history is the one without any customer. For firm 2, any "worst" history that is consistent with it playing conditionally undominated strategies and not exiting by time $2\tau^*(\lambda_G^2)$ involves observing a customer "right after" $\tau^*(\lambda_G^2)$. (See Figure 5.)

Fourth, combining the inference from these two private histories yields a lower bound to firm 1's posterior belief at time $2\tau^*(\lambda_G^2)$ along any history on the path generated by a conditionally undominated strategy. For convenience, we further approximate this bound conditioning only on $\omega_{\tau^*(\lambda_G^2)} = G$ and firm 1 not having observed any customers in $[\tau^*(\lambda_G^2), 2\tau^*(\lambda_G^2))$ to obtain the bound identified with the first red x mark in Figure 5.

Fifth, we can derive a lower bound on the belief of firm 1 at any time after $2\tau^*(\lambda_G^1)$ using the fact that the posterior belief cannot decrease faster than the private belief, that is, the belief computed while disregarding observational learning. (See footnote 26.) As it turns out, using this bound, we can show that exiting is a dominated action for firm 1 at any time before $\tau^*(\lambda_G^1) + \tau^*(\lambda_G^2)$. (See the red line in Figure 5.)

Last, at time $3\tau^*(\lambda_G^2)$ the same dominance arguments apply, because the problem is stationary. The stationarity hinges on the conclusive news assumptions, as if no firm exits before $3\tau^*(\lambda_G^2)$, it becomes common knowledge that $\omega_{2\tau^*(\lambda_G^2)} = G$. More precisely, for any $n = 3, 4, \ldots$ firm 2 finds it dominant to exit at any $t \in [(n-1)\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2))$ as soon as its belief hits $\pi^*(\lambda_G^2)$. Therefore, firm 1 finds it dominant not to exit at $t \in [n\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2) + \tau^*(\lambda_G^1))$, irrespective of its private history.

In the case of inconclusive news, i.e., $\lambda_B > 0$, computing the lower bound for the posterior belief is more complicated since it is unclear which pair of firms' private histories would give rise to the lowest posterior belief. Nevertheless, leveraging the

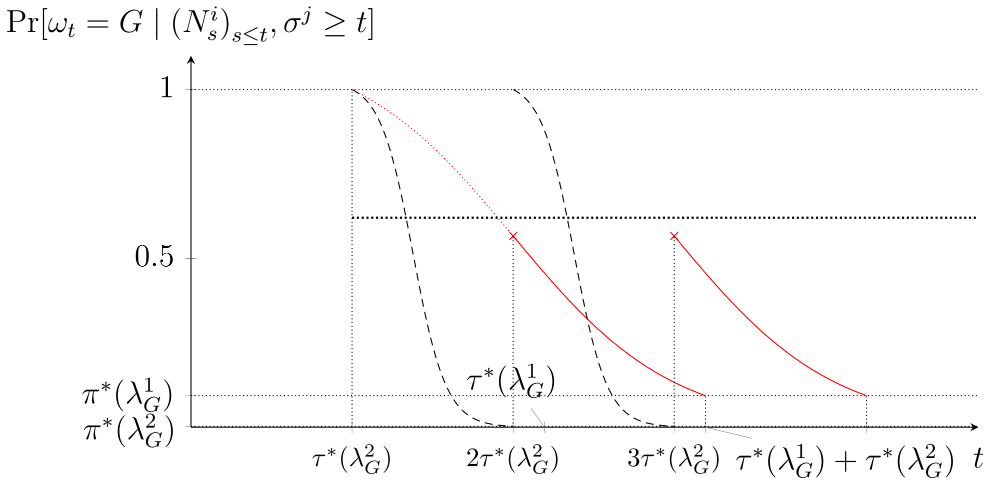


Figure 5: Illustration of the proof of Theorem 2. The black dashed line indicates the belief of firm 2 along an history with one customer at $\tau^*(\lambda^2_G)$ and one customer at $2\tau^*(\lambda^2_G)$. In red, the lower bound on firm 1's belief. In the figure, $(c, R, r, \gamma, \lambda^1_G, \lambda^2_G) = (1, 1, 1/10, 1/5, 3/2, 8)$.

continuity of the posterior belief in λ_B, we show that for any $\lambda_B > 0$, as λ^2_G goes to infinity, at any time along a history in which firm 2 does not exit, the posterior belief of firm 1 conditional on firm 2 playing undominated strategies is bounded away from the cutoff belief $\pi^*(\lambda^1_G)$.[27] Intuitively, as λ^2_G grows large, firm 1 bases its inference mostly on observational learning, and in the limit the fact that it learns via inconclusive bad news, instead of conclusive bad news, becomes irrelevant.

As noted, the two parts of Theorem 2 are not specular. The key step in the proof is to show that firm 2 finds it dominant to exit at $2\tau^*(\lambda^2_G)$ whenever its belief falls short of the cutoff $\pi^*(\lambda^2_G)$. Now, when the customer arrival rate of firm 2 is sufficiently large, its cutoff belief $\pi^*(\lambda^2_G)$ is arbitrarily low. As a result, in the limit, firm 2 finds it conditionally dominant to exit for "almost any" $\tau^*(\lambda^1_G) > 2\tau^*(\lambda^2_G)$. In contrast, when firm 2 is smaller than firm 1, i.e., $\lambda^2_G < \lambda^1_G$, firm 2 finds it conditionally dominant to exit at $2\tau^*(\lambda^2_G)$ only if $\tau^*(\lambda^1_G) - 2\tau^*(\lambda^2_G)$ is sufficiently large. That is, it is not firm 2's pessimism about the state of the world that determines its incentives to exit but rather the amount of time it expects the duopoly to last. The maximum first exit time is increasing in the operating cost c. As a result, if firm 2 is sufficiently impatient

[27] To be clear, the first part of Theorem 2 is valid for any $\lambda_B > 0$. We expect that the second part is valid for sufficiently low λ_B.

and the operating cost is such that the first exit time of firm 1 is sufficiently high, firm 2 will find it dominant to exit at $2\tau^*(\lambda_G^2)$.

4.4 Extensions

Other Asymmetries. Other types of asymmetries can easily be accommodated. Specifically, our results hold true if firms are asymmetric in their cost of operation (c), their revenues (R), their discount rate (r), or their rate of arrival of customers in the bad state (λ_B). More formally, for any set of parameters, $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$ and any λ_G^1, firm 2 exits first in the unique equilibrium of the game provided that λ_G^2 is large enough. In addition, for any set of parameters, $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$, there exists an open set of pairs $(\lambda_G^1, \lambda_G^2)$ such that $\lambda_G^2 < \lambda_G^1$ and firm 2 exits first in the unique equilibrium of the game provided that R^1/c^1, r^2, and γ are high enough.

In a setup with asymmetric primitive parameters, the following comparative statics result is almost immediate.

Proposition 2. *For any set of parameters $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$, the set of pairs $(\lambda_G^1, \lambda_G^2)$ identified in Theorem 2 is increasing (in the inclusion order) in r^2 and c^2.*

We believe also that the set of strategy profiles that survive iterated deletion of dominated strategy should be increasing in r^2 and c^2. Intuitively, for any strategy of firm 1, the expected continuation payoff of firm 2 along any history is decreasing in c^2 and r. Hence, in the iterated procedure, whenever along some history exiting is dominant for firm 2 in a game in which the operating cost of firm 2 is c^2 (the discount rate of firm 2 is r^2), exiting along that history is also dominant in a game in which the operating cost of firm 2 is $c^{2\prime} > c^2$ (the discount rate of firm 2 is $r^{2\prime} > r^2$). Deleting more strategies for firm 2 among those that prescribe remaining in the market along some history, makes firm 1 more optimistic and can only increase the histories along which remaining in the market is dominant. While intuitive, this argument is difficult to formalize unless one focuses on a specific deletion procedure, as we do in Proposition 2.

Numerical simulations suggest that the comparative static with respect to the discount rate also holds in a setup with symmetric primitive parameters. (See also Figure 4.)

Public Information. Our results are to some extent robust to the introduction of background public learning. For example, in the presence of public conclusive good news, as long as the rate of arrival of public news is low enough, if the customer arrival rate of firm 2 is sufficiently high, in the unique equilibrium, firm 2 exits first with probability one. On the one hand, along any history, the additional information from the public news is of little help to firm 2 in drawing an inference about the state of the world. On the other hand, if the informativeness of the public signal is sufficiently low, the bad news from the absence of public good news cannot offset the good news from firm 2 remaining in the market. As a result, our dominance argument holds true. However, as private learning plays a key role in our proof, a sufficiently informative public signal may overturn our result by weakening the role of signaling.

Alternately, consider the case in which the state is publicly revealed at the jump times of a (state-independent) Poisson process. In this case, as soon as the public signal reveals that the market has become unprofitable, the game enters a war of attrition phase, as in Section 3.1. However along the path with no bad conclusive news, the same analysis applies. Again, the uniqueness results hold with no substantial changes.

Other Signal Structures. While a complete analysis under an alternative signal structure is beyond the scope of this paper, it is worth noting that the good news assumption is crucial only insofar as it allows us to use a recursive argument to prove equilibrium uniqueness. A mixed-news conclusive signal structure, as the one above, would preserve this property, and, as long as the informativeness of the private signal is proportional to the expected profit flow, the first exit time would be single-peaked.

It is natural to wonder whether the results extend to Brownian signal structure. Brownian learning would introduce a few complications. For example, in the case of arithmetic Brownian motion like in Bergemann and Välimäki (2000), a firm can become arbitrarily pessimistic arbitrarily fast. Because the first exit time is virtually zero for both of the firms, clearly this is not the relevant measure of firms strength. Moreover, calculating and bounding first- and second- order beliefs requires overcoming different challenges, as compared to the case of exponential learning. Broadly speaking, we believe that the idea of providing bounds on private beliefs and proceeding by iterated deletion of conditionally dominated strategy could still be applicable.

Reversible Exit. Allowing for reversible exit complicates the equilibrium analysis, as firms now have access to a richer action space to signal their private information. The full characterization of equilibria in this case seems out of reach. However, whenever either of the two candidates strategy profile above is an equilibrium of the game with irreversible exit, it remains a Perfect Bayesian equilibrium of the game with reversible exit.

In fact, if news is inconclusive and being a monopolist is always profitable regardless of the state of the world, signaling plays a limited role, and regardless of the specification of the beliefs off the path, there exists an equilibrium in which the weaker firm exits first with probability one. If instead a monopolist incurs a loss whenever the industry is unprofitable, one can specify the (off-path) posterior of the stronger firm after re-entry of the rival so to deter its exit and sustain the equilibrium behavior.

Pricing. A few papers, such as Roberts (1986), have extended the Milgrom and Roberts (1982) limit-pricing model to capture post-entry predation. Conceivably, a firm may try to use its pricing strategy to convey some bad news about the market's profitability.[28] While we leave the analysis of dynamic private learning and signalling to future research, in this section we discuss a simple way to introduce pricing in our model in the spirit of Diamond (1971).

Suppose that at each moment in time each firm also set its price. A consumer of firm i arriving at time t observes firm i's price at that time, and decides whether to purchase from firm i or to incur a cost to inspect the price posted by the other firm.[29] Assume that consumers' willingness to pay is distributed according to some distribution function F, irrespective of ω_t and that $p(1 - F(p))$ is single-peaked. We now argue that regardless of whether firms observe each other's posted prices, the strategy profile in Theorem 1 is part of an equilibrium in which both firms charge the monopoly price $\arg\max_p p(1 - F(p))$.

Assume first that the posted price is unobservable to the competitor. First, in line with the Diamond's paradox, if customers expect both firms to charge the same price, they won't find it worthwhile to pay the search cost. Second, the weaker firm has no

[28]We are not aware of pricing models with two-sided private learning about a common underlying state of the world. The paper by Sweeting et al. (2019) is an exception: they extend their finite-horizon model to allow for this possibility.

[29]Agents are short-lived, i.e., cannot delay their purchase.

incentive to raise the price and lose a customer as this would not affect the behavior of its competitor, even if it would make it more optimistic about the market condition. The stronger firm has no incentives to lose customers to his competitor either because if anything, it would delay the time the weaker firm exits. Last, even if firms could obverse each other's price, the equilibrium can be sustained by an appropriate choice of off-path beliefs.

5 Interpretation as an Investment Game

While the main body of the paper focuses on exit from a declining industry, in this section we show that our model can be used to analyze entry decisions in the presence of private learning. As explained below, the equivalent entry model can offer insights into the dynamics of investment in a disruptive technology (Christensen, 1997). This section is written in a relatively informal way, as the objective is to illustrate the versatility of our model.

In the entry game, two firms operating in an established market have the option to irreversibly enter a market of unknown profitability (and simultaneously exit the established market). Operating in the new market yields to firm i a flow profit $\pi^i_{\omega_t}$ irrespective of the market structure, where ω_t denotes the profitability of the new market. The idea is that in the emerging market, competition is less fierce: either the new market can accommodate both firms, or neither of them. When two firms operate in the established market, each of them earns a flow profit D^i. Firm i obtains a flow profit M^i from being the monopolist in the established market, $M^i > D^i$. As usual, firms discount future payoffs at rate $r > 0$.

The profitability of the new market evolves over time. It takes two values, $\omega_t \in \{\mathcal{G}, \mathcal{B}\}$. Initially, both firms attach probability one to the market being unprofitable, $\omega_0 = \mathcal{B}$. In line with this interpretation, let $\pi^i_{\mathcal{G}} > \pi^i_{\mathcal{B}}$ and $\pi^i_{\mathcal{G}} > D^i > \pi^i_{\mathcal{B}}$, so that entering the new market is ever optimal. Conditional on being initially unprofitable, the market irreversibly becomes profitable, unbeknown to the two firms, at some random time that is exponentially distributed with parameter $\gamma > 0$. Before entering the new market, each firm learns about the quality of the market from conclusive bad news signals with intensity $\lambda^i_{\omega_t}$, where $\lambda^i_{\mathcal{B}} > \lambda^i_{\mathcal{G}} = 0$, $i = 1, 2$.

Assumption 2. *The followings hold for* $i = 1, 2$

$$\pi^i_{\mathcal{G}} = D^i + \frac{\kappa}{\lambda^i_{\mathcal{B}}}, \qquad \pi^i_{\mathcal{B}} = \frac{\kappa}{\lambda^i_{\mathcal{B}}}, \qquad M^i = D^i + \frac{\eta}{\lambda^i_{\mathcal{B}}},$$

where $\eta > \kappa > 0$.

To interpret the model and the assumption, consider the decision to invest in a disruptive technology. For example, minicomputers or desktop personal computers represented a disruptive technology to the mainframe computer producers in the seventies and eighties; similarly, digital imagery disrupted the analogue photography industry. In this context, Assumption 2 implies that the firm that is better at spotting disruptive opportunities is also the one that is better positioned in the established market. In line with the celebrated "innovator's dilemma," (Christensen, 1997) investing in the disruptive technology comes at a higher cost for the leader in the mainstream market because it cannibalizes its existing, profitable business. Nevertheless, the leader in the mainstream market may have superior information about the disruptive technology. Case in point: Kodak developed and patented many of the components of digital imaging technology but failed to make the transition from film to digital photography.[30]

While restrictive, the assumption ties together a firm's learning speed to its gain from investing in the new market.[31] We want to show that this entry game is "equivalent" to the exit game with parameters $\lambda^i_G = \lambda^i_{\mathcal{B}}$, $\lambda_B = 0$, $R^i = D^i$, and $c = \kappa$. In fact, in the entry model, given a strategy profile (σ^1, σ^2), the payoff of firm i can be written as:

$$\mathbf{E}^{(\sigma^1,\sigma^2)}\left[\int_0^{\sigma^i} e^{-rt}\left(D^i + \mathbf{1}_{\{\sigma^j<t\}}(M^i - D^i)\right)\mathrm{d}t + \int_{\sigma_i}^{\infty} e^{-rt}\pi^i_{\omega_t}\mathrm{d}t\right].$$

Subtracting the constant term $\mathbf{E}\left[\int_0^\infty e^{-rt}\left(\pi^i_{\omega_t}\right)\mathrm{d}t\right]$, we obtain

$$\mathbf{E}^{(\sigma^1,\sigma^2)}\left[\int_0^{\sigma^i} e^{-rt}\left(\left(D^i - \pi^i_{\omega_t}\right) + \mathbf{1}_{\{\sigma^j<t\}}(M^i - D^i)\right)\mathrm{d}t\right].$$

[30] See, for example, "Kodak's First Digital Moment," *The New York Times*, August 12, 2015: https://nyti.ms/2kanhht.

[31] The assumption reduces the degrees of freedom in choosing the parameters of the entry model. It simplifies the proof of the equivalence between this game and the game in the exit game. We believe that it could be easily relaxed.

Using the relationships above to replace the parameters of the exit game for the parameters of the entry game, the integral above is "almost proportional" to equation (1). The difference is that in the exit game, the payoff after the rival acts depends on the underlying state of the world, which is not true for the entry game. However, the behavior of a firm after its competitor exits plays a limited role in the analysis. In fact, it is almost immediate that Theorem 1 and Theorem 2.B generalize to this setup. Theorem 2.A holds in a slightly weaker form: for any set of parameters, there exists a set of pairs $(\lambda^1_{\mathcal{B}}, \lambda^2_{\mathcal{B}})$, $\lambda^2_{\mathcal{B}} > \lambda^1_{\mathcal{B}}$, under which, in the unique equilibrium, firm 2 invests first with probability one.[32]

With this interpretation, our model predicts that when the prospects of the disruptive technology are sufficiently uncertain, the firm that is better positioned in the established market fails to invest in it in a timely manner. On the other hand, the model predicts that a firm with a large informational comparative advantage will successfully seize the opportunity to establish itself in the new market. For example, IBM, which dominated the mainframe market, correctly assessed that it was worth investing in the PC market (see, for example, Christensen, 1997, Chapter 5).

Our entry model is certainty too parsimonious to describe the decision to invest in a disruptive technology, as it cannot account, for example, for the possibility of acquiring a competitor. Yet, the vast literature on disruptive technology seems to have overlooked the role of observational learning, which, as shown by our model, may affect the investment dynamics.

6 Conclusions

We analyze a dynamic model of exit from a stochastically declining market. We investigate how private learning affects the equilibrium dynamics. We provide sufficient conditions under which the equilibrium is unique. When the equilibrium is unique, the firm that we identify to be the weaker one exits first with probability one. We show that in our model, the strength of a firm is determined by its first exit time, a measure of its signaling disadvantage. Crucially, the first exit time is non-monotonic in the firm size or, equivalently, in the learning speed. As a result, our model provides a novel explanation, based on informational externalities, of the fact that in some industries, smaller firms survive the decline. Specifically, our paper offers a

[32] For the formal statements, see the Section OA.3 in the Online Appendix.

testable theory of exit that ties the industry economics primitives to exit dynamics. Furthermore, we show that the model can offer insight into the dynamics of entry in new market.

We conjecture that our proof technique can be applied to other asymmetric timing games with private learning, potentially with more than two players. For example, the recursive dominance argument can be adapted to show uniqueness in some asymmetric preemption games with evolving state and private learning by identifying an appropriate bound on a player's continuation payoff if he does not act at the single-agent optimal cutoff.[33]

Our results do not immediately extend a setup in which the market profitability fluctuates over time. This generalization may have the potential to provide a novel model of shakeouts. We leave these questions for future research.

[33] Thomas (2020) proves equilibrium uniqueness in two-player preemption games with private learning (and perfectly persistent state) using a different technique.

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A Omitted Proofs

A.1 Proofs for Section 4

Proof of Lemma 1. If $\lambda_G^i R \leq c$, even when the prevailing state is good, firm i's expected flow payoff is non-positive. Hence, the firm finds it optimal to exit immediately, i.e., $\pi^*(\lambda_G^i) = 1$. To prove the converse, we show that if $\lambda_B = 0$, whenever $\lambda_G^i R > c$, $\pi^*(\lambda_G^i) < 1$. A fortiori $\pi^*(\lambda_G^i) < 1$ when $\lambda_B > 0$.

When $\lambda_G^i R > c$, the value function of the best-reply problem solves the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} rv(p) = \big(p\lambda_G^i + (1-p)\lambda_B\big)\left(R + v\left(j\left(p\right)\right) - v(p)\right) - c \\ - \big(p(1-p)(\lambda_G^i - \lambda_B) + p\gamma\big)\, v'(p), \end{aligned} \tag{2}$$

where

$$j(p) = \frac{p\,\lambda_G^i}{p\,\lambda_G^i + (1-p)\lambda_B},$$

denotes the belief after observing a customer.

In the case of conclusive news, i.e., $\lambda_B = 0$, by solving the Hamilton-Jacobi-Bellman equation, we find that in the continuation region,

$$v(p) = -\left(1 - \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}\,p\right)\frac{c}{r} + \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}\,p\left(R + v(1)\right) + p\,\Omega(p)^{1+\frac{r}{\gamma+\lambda_G^i}}\,C,$$

where C is a constant of integration. Since

$$v(1) = -\left(\frac{\gamma + r}{\gamma + \lambda_G^i + r}\right)\frac{c}{r} + \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}\left(R + v(1)\right) + \gamma^{1+\frac{r}{\gamma+\lambda_G^i}}\,C,$$

we have

$$C = \gamma^{-1-\frac{r}{\gamma+\lambda_G^i}}\left(\left(\frac{\gamma + r}{\gamma + \lambda_G^i + r}\right)\left(v(1) + \frac{c}{r}\right) - \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}R\right). \tag{3}$$

Using smooth-pasting condition at the cutoff $\pi^*(\lambda_G^i)$, $v'(\pi^*(\lambda_G^i) = 0$,

$$\begin{aligned} v(1) = -\frac{c}{r} + \frac{\lambda_G^i}{\gamma + r}R \\ + \frac{\gamma + \lambda_G^i + r}{\gamma + r}\frac{\pi^*(\lambda_G^i)\lambda_G^i\gamma}{(\gamma + r)(\pi^*(\lambda_G^i)\lambda_G^i + r)\left(\frac{\Omega(\pi^*(\lambda_G^i))}{\gamma}\right)^{\frac{r}{\gamma+\lambda_G^i}} - \pi^*(\lambda_G^i)\lambda_G^i\gamma}R, \end{aligned}$$

where

$$\Omega_i(p) = \frac{\gamma + (1-p)\lambda_G^i}{p}.$$

Using this equation to replace $v(1)$ in equation (3), we obtain

$$C = \gamma^{-\frac{r}{\gamma+\lambda_G^i}} \frac{\pi^*(\lambda_G^i)\lambda_G^i\gamma}{(\gamma+r)(\pi^*(\lambda_G^i)\lambda_G^i + r)\left(\frac{\Omega(\pi^*(\lambda_G^i))}{\gamma}\right)^{\frac{r}{\gamma+\lambda_G^i}} - \pi^*(\lambda_G^i)\lambda_G^i\gamma} R,$$

which, replaced in the value matching condition, $v(\pi^*(\lambda_G^i)) = 0$,

$$-\left(1 - \frac{\lambda_G^i}{\gamma+\lambda_G^i+r}\,\pi^*(\lambda_G^i)\right)\frac{c}{r} + \frac{\lambda_G^i}{\gamma+\lambda_G^i+r}\,\pi^*(\lambda_G^i)\,(R+v(1)) \\ + \pi^*(\lambda_G^i)\,\Omega(\pi^*(\lambda_G^i))^{1+\frac{r}{\gamma+\lambda_G^i}}\,C = 0,$$

yields, after some manipulations,[34]

$$\Omega_i(\pi^*(\lambda_G^i)) = \frac{\lambda_G^i(\gamma+\lambda_G^i)(\gamma+\lambda_G^i+r)}{(\gamma+r)}\frac{R}{c} - \frac{\lambda_G^i(\gamma+\lambda_G^i+r)}{r} \\ + \frac{\gamma\lambda_G^i(\gamma+\lambda_G^i)}{r(\gamma+r)\left(\Omega_i(\pi^*(\lambda_G^i))/\gamma\right)^{r/(\gamma+\lambda_G^i)}}. \tag{4}$$

The right-hand side is decreasing in $\pi^*(\lambda_G^i)$, and the left-hand side is increasing in $\pi^*(\lambda_G^i)$. Hence, there exists at most one root of equation (4). At $\pi^*(\lambda_G^i) = 1$, $\lambda_G^i R > c$ implies that the left-hand side if smaller than the right-hand side, while in the limit at $\pi^*(\lambda_G^i)$ goes to zero, the opposite is true. It follows that there exists a unique root $\pi^*(\lambda_G^i) < 1$. Optimality follows by standard verification arguments. (See, for example Øksendal and Sulem, 2019, Theorem 3.2.).

In the general case, i.e. $\lambda_B \geq 0$ the existence and uniqueness results for functional differential equations guarantee that there exists a unique twice continuously differentiable solution to the Hamilton-Jacobi-Bellman equation given an initial guess for $v(1)$; see for example Corduneanu et al., 2016, Theorem 2.4.[35] Define the mapping $\Gamma : [0, (\lambda_G^i R - c)/r] \to \mathbf{R}$, which maps an initial guess $v(1)$ to the following function of the corresponding solution $\min_{p\in(0,1]} v(p) + |v'(p)|$. Notice that $\Gamma(0) = (\lambda_G^i R - c)/r$, while $\Gamma((\lambda_G^i R - c)/r) < 0$. By Corduneanu et al., 2016, Theorem 3.6, the mapping Γ is continuous. Hence, by the intermediate value theorem, there exists a guess such

[34] It can be checked that when $\gamma = 0$, (4) reduces to the cutoff characterization of Décamps and Mariotti (2004) and Keller et al. (2005).

[35] After a change of variable $q = 1 - p$, the functional differential equation (2) is a Volterra operator. After bounding the domain to $q \in [0, 1-\varepsilon)$, for arbitrarily small $\varepsilon > 0$, the assumptions of Corduneanu et al., 2016, Theorem 2.4 are satisfied.

that the solution to the Hamilton-Jacobi-Bellman equation satisfies $v(p) = v'(p) = 0$ for some $p \in (0,1)$. Again, optimality and uniqueness follow by standard verification arguments.

Using the value matching and smooth-pasting conditions, it is easily verified that the optimal cutoff $\pi^*(\lambda_G^i)$ satisfies the following equation

$$\pi^*(\lambda_G^i) = \frac{c}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)} - \frac{\lambda_B}{\lambda_G^i - \lambda_B}. \tag{5}$$

Given an increasing function v, the equation has at most one solution $\pi^*(\lambda_G^i)$. The right-hand side is decreasing in v, in j, and in λ_G^i. Both the value function v and the function j are increasing pointwise in λ_G^i. Hence, by the implicit function theorem,[36] we can conclude that the optimal cutoff $\pi^*(\lambda_G^i)$ is decreasing in λ_G^i.

(ii) In an abuse of notation, in the general case, i.e., $\lambda_B \geq 0$, we write

$$\Omega(p) = \frac{\gamma + (1-p)(\lambda_G^i - \lambda_B)}{p}.$$

First, as proved above, $\pi^*(\lambda_G^i)$ is decreasing in λ_G^i. Second, notice that

$$\tau^*(\lambda_G^i) = \frac{\ln\left(\Omega(\pi^*(\lambda_G^i))/\gamma\right)}{\gamma + \lambda_G^i - \lambda_B}.$$

It follows from equation (5) that

$$\pi^*(\lambda_G^i) \geq \frac{c}{(\lambda_G^i - \lambda_B)\left(R + \left(\lambda_G^i R - c\right)/r\right)} - \frac{\lambda_B}{\lambda_G^i - \lambda_B} =: \underline{\pi}(\lambda_G^i),$$

where the bound is derived by replacing $v(j(\pi^*(\lambda_G^i)))$ with $\left(\lambda_G^i R - c\right)/r$. Clearly, $v(j(\pi^*(\lambda_G^i))) < \left(\lambda_G^i R - c\right)/r$ because $\gamma > 0$ and the state market conditions eventually deteriorate.

By de l'Hôpital's rule

$$\begin{aligned}\lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\underline{\pi}(\lambda_G^i))/\gamma\right)}{\gamma + \lambda_G^i - \lambda_B} &= \lim_{\lambda_G^i \to \infty} \frac{1}{\Omega(\underline{\pi}(\lambda_G^i))}\left(\frac{1-\underline{\pi}(\lambda_G^i)}{\underline{\pi}(\lambda_G^i)} + \frac{\gamma + \lambda_G^i - \lambda_B}{\underline{\pi}(\lambda_G^i)^2}\underline{\pi}'(\lambda_G^i)\right) \\ &= \lim_{\lambda_G^i \to \infty} \frac{1-\underline{\pi}(\lambda_G^i)}{\gamma + (1-\underline{\pi}(\lambda_G^i))(\lambda_G^i - \lambda_B)} + \frac{\gamma + \lambda_G^i - \lambda_B}{\gamma + (1-\underline{\pi}(\lambda_G^i))(\lambda_G^i - \lambda_B)}\frac{\underline{\pi}'(\lambda_G^i)}{\underline{\pi}(\lambda_G^i)}.\end{aligned}$$

[36] To be precise, abusing notation let $v(p, \lambda_G^i)$ and $j(p, \lambda_G^i)$ be the optimal value function and the function that describes the belief after the arrival of a customer respectively, when the rate of arrival of consumers in the good state is λ_G^i. Then, $v(p, \lambda_G^i)$ is continuously differentiable in p, and $j(p, \lambda_G^i)$ is continuously differentiable in λ_G^i. By the envelope theorem of Milgrom and Segal (2002), the value function $v(p, \lambda_G^i)$ is continuously differentiable in λ_G^i.

Since $\underline{\pi}'(\lambda_G^i) = O((\lambda_G^i)^{-2})$ and $\underline{\pi}(\lambda_G^i) = O((\lambda_G^i)^{-1})$, the right-hand side converges to zero. Hence,

$$0 = \lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\underline{\pi}(\lambda_G^i)/\gamma)\right)}{\gamma + \lambda_G^i - \lambda_B} \geq \lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\pi^*(\lambda_G^i)/\gamma)\right)}{\gamma + \lambda_G^i - \lambda_B},$$

and $\lim_{\lambda_G^i \to \infty} \tau^*(\lambda_G^i) = 0$. We now state and prove two lemmas that are used later.

Lemma 3. *The following holds,*

$$\lim_{\lambda_G^i \to \infty} \pi^*(\lambda_G^i)\lambda_G^i \to 0.$$

Proof. First, in the case of conclusive news, i.e., $\lambda_B = 0$, smooth-pasting condition implies

$$0 = \lambda_G^i \pi^*(\lambda_G^i)(R + v(1)) - c.$$

Moreover,

$$v(1) \geq \frac{\lambda_G^i \left(1 - e^{(\gamma+\lambda_G^i+r)\tau}\right)}{\gamma + r + \lambda_G^i e^{(\gamma+\lambda_G^i+r)\tau}} R - \frac{(\gamma + r)(\lambda_G^i + \gamma) - \lambda_G^i r e^{-(\gamma+\lambda_G^i+r)\tau} + \gamma(r + \gamma + \lambda_1)e^{-r\tau}}{r(\gamma + \lambda_G^i)\left(\gamma + r + \lambda_G^i e^{(\gamma+\lambda_G^i+r)\tau}\right)} c$$

for all $\tau \geq 0$. Hence, $v(1) \to \infty$ as $\lambda_G^i \to \infty$. As a result, $\lim_{\lambda_G^i \to \infty} \lambda_G^i \pi^*(\lambda_G^i) = 0$. Because the cutoff belief in the case of conclusive news is an upper bound to the cutoff belief in the case of inconclusive news, it follows that the result generalizes to the case of $\lambda_B > 0$. □

Lemma 4.

(i) $\lim_{\lambda_G^i \to \infty} \pi^{*\prime}(\lambda_G^i) = 0.$

(ii) $\lim_{\lambda_G^i \to \infty} \pi^{*\prime\prime}(\lambda_G^i) = 0,$

Proof. (i) As in footnote 36, we let $v(p, \lambda_G^i)$ and $j(p, \lambda_G^i)$ be the optimal value function and the function that describes the belief after the arrival of a customer, respectively, when the rate of arrival of consumers in the good stat e is λ_G^i. By the implicit function theorem,

$$\begin{aligned}\pi^{*\prime}(\lambda_G^i) = &- \frac{\pi^*(\lambda_G^i)}{\lambda_G^i - \lambda_B} - \frac{c}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right)\right)^2} \\ &\Bigg(v_p\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right) \frac{\pi^*(\lambda_G^i)(1 - \pi^*(\lambda_G^i))\lambda_B}{(\pi^*(\lambda_G^i)\lambda_G^i + (1 - \pi^*(\lambda_G^i))\lambda_B)^2} \\ &+ v_{\lambda_G^i}\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right)\Bigg).\end{aligned} \tag{6}$$

By Lemma 3, $\pi^*(\lambda_G^i)\lambda_G^i \to 0$; it follows that $\lim_{\lambda_G^i \to \infty} \pi^{*\prime}(\lambda_G^i) = 0$.

(ii) Denote the term in parentheses in equation (6) by $D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$ and its derivative with respect to λ_G^i by $D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$.

By the implicit function theorem,

$$\begin{aligned}\pi^{*\prime\prime}(\lambda_G^i) = &\frac{2\pi^*(\lambda_G^i)}{(\lambda_G^i - \lambda_B)^2} + \left(\frac{2\left(\pi^*(\lambda_G^i)\lambda_G^i + (1 - \pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)}\right) D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right) \\ &+ \frac{2\left(\pi^*(\lambda_G^i)\lambda_G^i + (1 - \pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)^2} \left(D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)\right)^2 \\ &- \frac{\left(\pi^*(\lambda_G^i)\lambda_G^i + (1 - \pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)} D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right).\end{aligned}$$

The term on the right-hand side converges to zero, as $D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$ is bounded. □

The proof of single-peakedness relies on the following lemma, which establishes a scale-invariance property of the optimal stopping problem.

Lemma 5. *Consider two sets of parameters* $(c, R, r, \lambda_G^i, \lambda_B, \gamma)$ *and* $(\hat{c}, R, \hat{r}, \hat{\lambda}_G^i, \hat{\lambda}_B, \hat{\gamma})$ *such that*

$$\frac{\hat{\lambda}_G^i}{\hat{\gamma}} = \frac{\lambda_G^i}{\gamma}, \qquad \frac{\hat{\lambda}_B}{\hat{\gamma}} = \frac{\lambda_B}{\gamma},$$

$$\frac{\hat{r}}{\hat{\gamma}} = \frac{r}{\gamma}, \qquad \frac{\hat{c}}{\hat{\gamma}} = \frac{c}{\gamma}.$$

The optimal value functions and the optimal cutoff beliefs in the two optimal stopping problems coincide.

Proof. First, notice that the optimal value function associated with the first optimal stopping problem satisfies the Hamilton-Jacobi-Bellman equation of the second; see (2). In addition, the smooth pasting and value matching conditions associated with the two optimal stopping problems are identical. Since a standard verification theorem applies (see Øksendal and Sulem, 2019, Theorem 3.2), the optimal value functions and the optimal cutoff beliefs in the two optimal stopping problems coincide. □

Throughout, we fix a vector (c, R, r, λ_B). In light of Lemma 5, we consider the optimal stopping problem "scaled" by some γ, that is, we should consider how the first exit time changes with λ_G^i in the optimal stopping problem parametrized by $(\gamma c, R, \gamma r, \gamma\lambda_G^i, \gamma\lambda_B, \gamma)$, where the first element is the flow cost of remaining into business.

Define $\hat{\pi}^*(\lambda_G^i)$ to be the optimal cutoff when $\gamma = 1$, and the other parameters are $(c, R, r, \lambda_G^i, \lambda_B)$. By construction, the optimal cutoff associated with the decision problem parametrized by $(\gamma c, R, \gamma r, \gamma\lambda_G^i, \gamma\lambda_B, \gamma)$ is also $\hat{\pi}^*(\lambda_G^i)$. Define

$$\hat{\tau}^*(\Lambda_G^i; \gamma) = \frac{\ln\left(\frac{1}{\hat{\pi}^*(\Lambda_G^i)} + \frac{1-\pi^*(\Lambda_G^i)}{\pi^*(\lambda_G^i)}\left(\Lambda_G^i - \lambda_B\right)\right)}{\gamma(1 + \Lambda_G^i - \lambda_B)}$$

so that the first exit time associated with the decision problem parametrized by $(\gamma c, R, \gamma r, \gamma\lambda_G^i, \gamma\lambda_B, \gamma)$ is equal to $\hat{\tau}^*(\Lambda_G^i; \gamma)$. To say it differently, for any set of parameters, $\tau^*(\lambda_G^i) = \hat{\tau}^*(\lambda_G^i/\gamma; \gamma)$. Clearly, proving the single-peakedness of $\hat{\tau}^*$ is equivalent to proving the single-peakedness of τ^*.

By differentiation,

$$\begin{aligned}\hat{\tau}^{*\prime}(\Lambda_G^i; \gamma) =& \frac{1}{\gamma(1 + \Lambda_G^i - \lambda_B)}\Bigg(- \hat{\tau}^*(\Lambda_G^i; \gamma) \\ &+ \Bigg(\frac{1 - \hat{\pi}^*(\Lambda_G^i)}{1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B)} \\ &- \frac{1 + \Lambda_G^i - \Lambda_B}{(1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B))\,\hat{\pi}^*(\Lambda_G^i)}\hat{\pi}^{*\prime}(\Lambda_G^i)\Bigg)\Bigg).\end{aligned} \tag{7}$$

We now define

$$\psi(\Lambda_G^i) := \frac{1 - \hat{\pi}^*(\Lambda_G^i)}{1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B)} - \frac{1 + \Lambda_G^i - \Lambda_B}{(1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B))\,\hat{\pi}^*(\Lambda_G^i)}\hat{\pi}^{*\prime}(\Lambda_G^i).$$

Notice that $\psi(\Lambda_G^i)$ does not depend on γ. The following lemma establishes two properties of the function $\psi(\Lambda_G^i)$, which we use later.

Lemma 6.

(i) $\psi'(\Lambda_G^i) < 0$ *a.e. for* $\Lambda_G^i \geq \underline{\Lambda}_G^i$, *for some* $\underline{\Lambda}_G^i \in (c/R + 1/2, \infty)$.

(ii) $\inf_{\Lambda_G^i \in (c/R+1/2, \underline{\Lambda}_G^i]} \psi(\Lambda_G^i) > 0.$

Proof. For (i), differentiating yields

$$\psi'(\Lambda_G^i) = -\left(\frac{1}{1+(1-\hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i-\lambda_B)}\right)^2$$

$$\left(\left(1-\hat{\pi}^*(\Lambda_G^i)\right)^2 + 2\hat{\pi}^{*\prime}(\Lambda_G^i) + (1+\Lambda_G^i-\lambda_B)(1+(1-2\hat{\pi}^*(\Lambda_G^i))(\lambda_G^i-\lambda_B))\left(\frac{\hat{\pi}^{*\prime}(\Lambda_G^i)}{\hat{\pi}^*(\Lambda_G^i)}\right)^2\right)$$

$$-\frac{1+\Lambda_G-\lambda_B}{\hat{\pi}^*(\Lambda_G^i)\left(\gamma+(1-\hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i-\lambda_B)\right)}\hat{\pi}^{*\prime\prime}(\Lambda_G^i).$$

Recall that $\hat{\pi}^{*\prime}(\Lambda_G^i) < 0$ and $\lim_{\Lambda_G^i\to\infty}\hat{\pi}^*(\Lambda_G^i) = 0$. Moreover, by Lemma 4, for sufficiently high Λ_G^i, the terms on the second and third lines are positive a.e. for $\Lambda_G^i > \underline{\Lambda}_G^i$ for some $\underline{\Lambda}_G^i < \infty$. Because $\underline{\Lambda}_G^i < \infty$, result (ii) follows.

□

To conclude, we first show that $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is single-peaked for sufficiently high γ. Observe that $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is (pointwise in Λ_G^i) decreasing in γ. Hence, for sufficiently high γ, for all $\Lambda_G^i \in (c/R+1/2,\underline{\Lambda}_G^i]$, $\hat{\tau}^*(\Lambda_G^i;\gamma) < \inf_{\Lambda_G^i\in(c/R+1/2,\underline{\Lambda}_G^i]}\psi(\Lambda_G^i)$. Consequently, for γ high enough, the function $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is single-peaked as $\hat{\tau}^*(\Lambda_G^i;\gamma)$ crosses the function $\psi(\Lambda_G^i)$ no more than once and from below (see (7)). Therefore, $\hat{\tau}^*(\Lambda_G^i;\gamma)$ must be single-peaked for any γ, because a linear transformation of a single-peaked function is single-peaked.

□

Proof of Theorem 1. Here, we provide the proof of Theorem 1 for the case of inconclusive news, i.e., $\lambda_B > 0$.

With a slight abuse of notation, let $\tau^*(\lambda_2^G,\lambda_B)$ and $\pi^*(\lambda_2^G,\lambda_B)$ denote the first exit time and the cutoff belief in the benchmark best-reply problem as a function of the arrival rate in the two states.

We want to prove that for sufficiently high λ_2^G, the strategy profile $(\sigma_0^1,\sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium of the game. First, by Lemma 1, for sufficiently high λ_2^G, $2\tau^*(\lambda_2^G,\lambda_B) < \tau^*(\lambda_1^G,\lambda_B)$. Now, along the path induced by this strategy profile, if firm 2 has not exited by time $t>0$, the posterior likelihood about the prevailing state is bounded below by

$$\frac{e^{-(\gamma+\lambda_G^1)t}\Pr\left[\Pr\left[\omega_s = G \mid N_t^2\right] > \pi^*\left(\lambda_G^2,\lambda_B\right),\ \text{for all } s\le t \mid \omega_t = G\right]}{\displaystyle\int_0^t \gamma e^{-\gamma s}e^{-\lambda_G^1 s}e^{-\lambda_B(t-s)}\left(e^{-\lambda_B(t-s)}\sum_{i=\left\lfloor\frac{t-s}{\tau^*(\lambda_G^2,\lambda_B)}\right\rfloor}^{\infty}\frac{\left(\lambda_B^2(t-s)\right)^i}{i!}\left(1-e^{-\lambda_B\tau^*\left(\lambda_G^2,\lambda_B\right)}\right)^i\right)\mathrm{d}s}$$

This is a lower bound because at any time $s\le t$, firm 2 having observed at least $\lfloor s/\tau^*(\lambda_G^2)\rfloor$ customers, with the time between two customers being no larger than

$\tau^*(\lambda_G^2)$ is a necessary but not sufficient condition for it to remain in the market up to time t. Moreover at the denominator, we are writing the probability that firm 2 observed sufficiently many customers once the state has transitioned to bad, but we omit the probability of observing sufficiently many customers at earlier times when the state was still good.

As shown in Lemma OA.1 in the online Appendix we show that denominator converges to zero as $\lambda_G^2 \to \infty$, uniformly in t. It is easy to see that the numerator is always bounded away from zero. As a result, the posterior belief of firm 1 is again bounded away from the cutoff $\pi^*(\lambda_1^G, \lambda_B)$ and by Lemma 2, σ_0^1 is a best reply to $\sigma^2_{\pi^*(\lambda_G^2, \lambda_B)}$.

□

The proof of Theorem 2 relies on iterated deletion of conditionally dominated strategies. The notion of conditional dominance was introduced by Shimoji and Watson (1998). Informally, according to Shimoji and Watson's (1998) definition, a strategy is conditionally dominated if one can find an information set such that the strategy is strictly dominated when one restricts attention to strategies that are consistent with reaching that information set. Iterative deletion of conditionally dominated strategies is then defined as in the case of normal form games.

Shimoji and Watson (1998) recognize that their result analysis extends to games with incomplete information, but do not spell out this extension. In our model, we say that a strategy σ^i is conditionally dominated at some history if there exists another strategy $\hat{\sigma}^i$ which prescribes a different behavior at that history and potentially at some of its successors, and agrees with σ^i at any other history, and such that for any strategy σ^j which is consistent with that history and any system of beliefs consistent with Bayes' rule, $\hat{\sigma}^i$ yields a strictly higher expected continuation payoff than σ^i.

Proof of Theorem 2.A. We show that for sufficiently high λ_G^2, $(\sigma_0, \sigma_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of conditionally dominated strategies.

First, assume that $\lambda_B = 0$ and $2\tau^*(\lambda_G^2) < \tau^*(\lambda_G^1)$. By Lemma 1, this inequality holds for sufficiently high λ_G^2. Recall that by Lemma 2, in any equilibrium firm i continues operations as long as its belief is above $\pi^*(\lambda_G^i)$. As a result, firms' beliefs at any time before $\tau^*(\lambda_G^2)$ is uniquely determined by their private history.

We now argue that regardless of firm 1's belief about firm 2's strategy, firm 1's posterior along the history with no exit is bounded above $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$.

To this end, we start by showing that at any time $t \leq \tau^*(\lambda_G^1)$, for any strategy of firm 2 that survived our first round of deletion, the probability that firm 2's posterior belief is equal to some $p \in [0, \pi^*(\lambda_G^2))$ is higher in bad state than in the good state, provided that λ_G^2 is taken to be arbitrary high. This implies that along the history with no exit, observational learning always brings good news, and firm 1's private belief is a lower bound to its posterior belief. The details of the argument are relegated to the online Appendix, but here we provide some intuition.

If firm 1 expects firm 2 to play a cutoff strategy, as in Rosenberg et al. (2007) and Murto and Välimäki (2011), observational learning always brings good news, that is, firm 2 continuing operation makes firm 1 more optimistic. In fact, the distribution of firm 2's posterior belief conditional on the good state first-order stochastically dominates the distribution of firm 2's posterior belief conditional on the bad state. Once one allows for any non-cutoff strategy, this does need to be true. However, in the limit as λ_G^2 goes to infinity, two things happen. First, the distribution of firm 2's posterior beliefs conditional on either states converges to a degenerate distribution concentrated on either 0 or 1. Second, the range of belief for which exiting is not a dominated action shrinks, since $\pi^*(\lambda_G^2) \to 0$. As a result, even if firm 1 expects firm 2 to play a non-cutoff strategy, the probability of firm 2 exiting conditional on the prevailing state being bad is higher than the probability of firm 2 exiting conditional on the state being good. It follows that exiting before $\tau^*(\lambda_G^1)$ is a dominated action for firm 1 and firm 2's belief at any time before $\tau^*(\lambda_G^1)$ is uniquely determined by its private history.

Consider the case in which firm 2 does not observe any customer in the interval $[0, \tau^*(\lambda_G^2))$. At time $\tau^*(\lambda_G^2)$, the expected continuation payoff of firm 2 from remaining in the market forever is bounded above by

$$\int_{\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2)e^{-\gamma(t-\tau^*(\lambda_G^2))}\lambda_G^2 R - c\right) \mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R - c}{r}\right).$$

The expression above is the continuation payoff of firm 2 at time $\tau^*(\lambda_G^2)$ in the hypothetical scenario in which firm 1 exits with probability one at time $\tau^*(\lambda_G^1)$. At that time, firm 2 perfectly learns the state of the world such that, conditional on $\omega_{\tau^*(\lambda_G^1)} = B$, it exits with no delay. Clearly, this is an upper bound to continuation payoff of firm 2 for any undominated strategy adopted by firm 1.

Observe that by the definition of $\tau^*(\lambda_G^2)$, the integrand is negative; thus the first term is negative. By Lemma 3, as $\lambda_G^2 \to \infty$, $\pi^*(\lambda_G^2)\lambda_G^2 \to 0$. Hence, for sufficiently high λ_2^G, the expected continuation payoff is negative. We can then conclude that conditional on not observing any customer in $[0, \tau^*(\lambda_G^2))$, it is dominant for firm 2 to exit at time $\tau^*(\lambda_G^2)$.

Consider now the case in which the posterior belief of firm 2 at time $2\tau^*(\lambda_G^2)$ is again $\pi^*(\lambda_G^2)$. In this case, the expected continuation continuation payoff from remaining in the market is bounded above by

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2 R - c\right) \mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right). \tag{8}$$

By the same argument as above, for sufficiently high λ_G^2, the expected continuation payoff is negative and exit is dominant for firm 2. A fortiori, for any $t \in (\tau^*(\lambda_G^2), 2\tau^*(\lambda_G^2)]$, it is dominant for firm 2 to exit whenever its belief falls short of $\pi^*(\lambda_G^2)$.

In the case of conclusive news, if firm 1 does not observe an exit at $2\tau^*(\lambda_G^2)$, it is dominant for it not to exit before $\tau^*(\lambda_G^1) + \tau^*(\lambda_G^2)$. In fact, at that time, firm 1 infers that the belief of firm 2 never fell below the cutoff $\pi^*(\lambda_G^2)$ in $[0, 2\tau^*(\lambda_G^1))$. Moreover, in the worst-case scenario, firm 2 observed a customer "right after" $\tau^*(\lambda_G^2)$ (see Figure 5). Consequently, the posterior belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)+\tau^*(\lambda_G^2)$, and by Lemma 1 remaining in the market is dominant at those times.

To show the desired result, we apply conditional dominance argument recursively. More formally, for any $n = 3, 4, \ldots,$ firm 2 finds it dominant to exit at any $t \in [(n-1)\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2))$ as soon as its belief falls short of $\pi^*(\lambda_G^2)$; in fact, for any n, the payoff from staying in the market is bounded above by

$$\int_{n\tau^*(\lambda_G^2)}^{(n-2)\tau^*(\lambda_G^2)+\tau^*(\lambda_G^1)} e^{-r(t-n\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2)e^{-\gamma(t-n\tau^*(\lambda_G^2))}\lambda_G^2 R - c\right) \mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right).$$

Again, for sufficiently high λ_G^2, this bound is negative and exiting is dominant for firm 2. Given this, firm 1 finds it dominant not to exit at all $t \in [(n-1)\,\tau^*(\lambda_G^2), (n-1)\,\tau^*(\lambda_G^2) + \tau^*(\lambda_G^1))$, irrespective of its private history.

For the case of inconclusive news, we can again apply the limit argument we used in the proof of Theorem 1 to show that $(\sigma_0, \sigma_{\pi^*(\lambda_G^2)})$ is the unique outcome that survives iterated deletion of conditionally dominated strategies. In this case, at time $2\tau^*(\lambda_G^2)$, the continuation payoff of firm 2 is bounded above by (omitting the dependence of

π^* and τ^* on λ_B)

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \left(\left(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2 + \left(1-\pi^*(\lambda_G^2,)\right)\lambda_B^2\right)R - c\right)\mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right).$$

Recall that this is the continuation payoff at time $2\tau^*(\lambda_G^2)$ in the hypothetical scenario in which firm 1 exits with probability one at time $\tau^*(\lambda_G^1)$; at that time, firm 2 perfectly learns the state and exits with no delay if $\omega_{\tau^*(\lambda_G^1)} = B$, because by assumption, $(\lambda_B^1 + \lambda_B^2)R < 0$. Again, the integrand is negative, and by Lemma 3, the second term converges to zero as $\lambda_G^2 \to \infty$. As a result, exiting is dominant for firm 2 at any time before $2\tau^*(\lambda_G^2)$ as soon as its beliefs fall short of $\pi^*(\lambda_G^2)$. Then, by the limit argument, for sufficiently high λ_G^2, the belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time in $[\tau^*(\lambda_G^1), \tau^*(\lambda_G^1)+\tau^*(\lambda_G^2))$. Hence, it is dominant for firm 1 not to exit before $\tau^*(\lambda_G^1)+\tau^*(\lambda_G^2)$. The remainder of the proof follows from the same recursive argument as in the previous part.

□

If being a monopolist is profitable in both states, that is, $(\lambda_B^1 + \lambda_B^2)R - c > 0$, then the relevant bound becomes

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \Bigg(\Bigg(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2 \\ + \left(1-\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\right)\lambda_B^2\Bigg)R - c\Bigg)\mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right) \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\left(1-e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\pi^*(\lambda_G^2)\right)\frac{2\lambda_B R - c}{r}\right).$$

Notice that as $\lambda_G^2 \to \infty$, $\tau^*(\lambda_G^2) \to 0$, and $\pi^*(\lambda_G^2)\lambda_G^2 \to 0$. Hence, in the limit, as $\lambda_G^2 \to \infty$, the bound converges to

$$\frac{1-e^{-r\tau^*(\lambda_G^1)}}{r}(\lambda_G^2 R - c) + \frac{e^{-r\tau^*(\lambda_G^1)}}{r}(2\lambda_B R - c). \tag{9}$$

From equation (5) (see also Lemma 7 below),

$$\tau^*(\lambda_G^1) \geq \frac{1}{\gamma^1 + \lambda_G^1 - \lambda_B}\ln\left(\frac{(\lambda_G^1 - \lambda_B)((\lambda_G^1+\gamma)R - c)}{\gamma(c - \lambda_B R)}\right).$$

Replacing $\tau^*(\lambda_G^1)$ with this bound in equation (9), we obtain

$$-\frac{c-\lambda_B R}{r}+\frac{\lambda_B R}{r}\left(\frac{(\lambda_G^1-\lambda_B)((\lambda_G^1+\gamma)R-c)}{\gamma(c-\lambda_B R)}\right)^{-\frac{r}{\gamma+\lambda_G^1-\lambda_B}}. \tag{10}$$

We now claim that there exists a set of λ_G^1 such that this bound is negative. Assume that $r>\gamma+\lambda_B$. Then, if $\lambda_G^1=2\lambda_B$, equation (10) can be shown to be strictly negative. By continuity, we can then conclude that there exists a set of pairs $\mathcal{L}\in(c/R,\infty)\times(c/R,\infty)$, $\lambda_G^2>\lambda_G^1$, for which $(\sigma_0^1,\sigma^2_{\pi^*(\lambda_G^2)})$ is the unique equilibrium.

Proof of Theorem 2.B. First, for any c and R, we can choose λ_G^2 arbitrarily close to c/R such that $2\tau^*(\lambda_G^1)<\tau^*(\lambda_G^2)$.

Second, we show that for firm 1, exiting before $\tau^*(\lambda_1^G)$ is a dominated action. As in Theorem 2.A, we argue that firm 1's posterior along the history with no exit is bounded above $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$. The formal proof is in the Online Appendix (see Section OA.2.2). Here we provide an informal argument.

We show that in the limit as $R/c\to\infty$ and $\gamma\to\infty$, for λ_G^2 appropriately chosen, firm 2 not exiting always brings good news to firm 1, regardless of which strategy firm 1 expects firm 2 to play, among the strategy surviving our first round of deletion. Intuitively, the inference firm 1 draws from observing the action of firm 2 always concerns the state at some point in time in the past, and not about the prevailing state. Hence, as $\gamma\to\infty$, this inference plays a limited role in determining firm 1's posterior belief, which can be shown to be bounded away from $\pi^*(\lambda_G^1)$ at any time before the first exit time $\tau^*(\lambda_G^1)$.

Third, proceeding as in the proof of Theorem 2.A, we show that (8) is negative provided that R/c and r sufficiently high. That is,

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))}\left(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2R-c\right)\mathrm{d}t \\ +e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right)<0.$$

Again, by the definition of $\pi^*(\lambda_G^2)$, the first term is negative. By Lemma 7, for sufficiently high R/c and $r>\lambda_G^1$, the second term converges to zero whenever λ_G^2 is chosen to be arbitrarily close to c/R. Crucially, the first integral in the equation above remains bounded away from zero as we take this limit because $\tau^*(\lambda_G^1)$ is increasing in R/c. Following the same steps as before, we can then prove that $(\sigma_0^1,\sigma^2_{\pi^*(\lambda_G^2})$ is the unique strategy profile that survives iterated deletion of dominated strategies. □

Lemma 7. *If* $r > \lambda_G^1$,

$$e^{-(r+\gamma)\tau^*(\lambda_G^1)}\frac{\lambda_G^1 R}{r} \to 0,$$

as $R \to \infty$ *or* $c \to 0$.

Proof. We are going to derive a lower bound for $\tau^*(\lambda_G^1)$ by identifying an upper bound for $\pi^*(\lambda_G^1)$. From (5), replacing $v(j(\pi^*(\lambda_G^i)))$ with 0, we obtain

$$\tau^*(\lambda_G^1) \geq \frac{1}{\gamma + \lambda_G^1} \ln\left(-\frac{\lambda_G^1}{r} + \frac{\gamma + \lambda_G^1}{\gamma} \cdot \frac{\lambda_G^1 R}{c}\right).$$

Hence,

$$e^{-(r+\gamma)\tau^*(\lambda_G^1)}\frac{\lambda_G^1 R}{r} \leq \left(-\frac{\lambda_G^1}{r} + \frac{\gamma + \lambda_G^1}{\gamma} \cdot \frac{\lambda_G^1 R}{c}\right)^{-\frac{\gamma+r}{\gamma+\lambda_G^1}} \frac{\lambda_G^1 R}{r}.$$

If $r > \lambda_G^1$, the right-hand side converges to 0 as $R \to \infty$ or $c \to 0$. In fact, the necessary condition for the right-hand side to converge to 0 is that as $R \to \infty$, R/c converges to infinity in the order $O(R)$. □

Proof of Proposition 2. We show that if (8) is negative for some c^2, then it is also negative for $c^{2\prime} > c^2$. Abusing notation, let $\tau^*(\lambda_G^2, c^2)$ and $\pi^*(\lambda_G^2, c^2)$ be the first exit time and the cutoff belief as a function also of the operating cost c^2. If

$$\begin{aligned}
&\int_{2\tau^*(\lambda_G^2,c^2)}^{\tau^*(\lambda_G^1,c^1)} e^{-r(t-2\tau^*(\lambda_G^2,c^2))}\left(\pi^*(\lambda_G^2,c^2)e^{-\gamma(t-2\tau^*(\lambda_G^2,c^2))}\lambda_G^2 R - c^2\right)\mathrm{d}t \\
&+ e^{-r\left(\tau^*(\lambda_G^1,c^1)-2\tau^*(\lambda_G^2,c^2)\right)}\left(\pi^*(\lambda_G^2,c^2)e^{-\gamma\left(\tau^*(\lambda_G^1,c^1)-2\tau^*(\lambda_G^2,c^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R - c^2}{r}\right) < 0,
\end{aligned}$$

then exiting at $2\tau^*(\lambda_G^2, c^2)$ following a history with no customer is dominant for firm 2 also when its cost is $c^{2\prime} > c^2$. Hence, if firm 1 does not observe an exit at $2\tau^*(\lambda_G^2, c^2)$, it is dominant for it not to exit before $\tau^*(\lambda_G^1, c^2) + \tau^*(\lambda_G^2, c^1)$, regardless of the operating cost of firm 2. Applying the logic recursively, we can delete all strategies of firm 1 but σ_0^1. Then, any strategy of firm 2 other than $\sigma^2_{\pi^*(\lambda_G^2,c^{2\prime})}$ can be deleted. It follows that the only strategy profile that survives iterated deletion of dominated strategy is $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2,c^{2\prime})})$. For the case of inconclusive news, the result follows from combining the argument above and the limit in the proof of Theorem 2.A.

Last, the proof of the comparative statics result with respect to r^2 follows from a similar argument. □

A.2 Proofs for Section 3.2

With some abuse of notation, we denote with σ_p^j the pure strategy that prescribes exiting as soon as the public belief about the prevailing state falls below a cutoff $p > 0$. Proceeding in the same way as in the previous Section 4.1, we start by analyzing firm i's best-reply problem to σ_0^j. In the continuation region, the value function of firm i, when best replying to σ_0^j satisfies the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} rv^i(p) = p\left(\lambda_G^i + \lambda_G^j\right)\left(v^i\left(j\left(p\right)\right) - v^i(p)\right) - c + p\lambda_G^i R \\ - \left(p(1-p)(\lambda_G^i + \lambda_G^j) + p\gamma\right) v^{i\prime}(p). \end{aligned}$$

As a result, whenever $\lambda_G^i > c/R$, the optimal cutoff of firm i satisfies the following equation

$$\mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = \frac{c}{\left(\lambda_G^i + \lambda_G^j\right) v^i\left(j(\mathring{\pi}^*(\lambda_G^i, \lambda_G^j))\right) + \lambda_G^i R} \tag{11}$$

Notice that in contrast to the case of unobservable customers, the optimal cutoff of firm i depends on both λ_G^i and λ_G^j. In fact, using the implicit function theorem, it is readily verified that $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j)$ is decreasing both in λ_G^i and in λ_G^j, and $\lim_{\lambda_G^i \to \infty} \mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = \lim_{\lambda_G^j \to \infty} \mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = 0$.

It can be shown that, again, $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j) < 1$ if and only if $\lambda_G^i R > c$.

Claim 1. *If $\lambda_G^2 > \lambda_G^1$, $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$.*

Proof. The value functions of the two firms are ranked pointwise, while the function j is identical for both of them. It follows that $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$. □

Proof of Proposition 1. By the same argument as in Lemma 2, in any equilibrium, it is dominant for firm i to exit whenever the belief is strictly above $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j)$. If $\lambda_G^2 > \lambda_G^1$, then $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$, and by the same argument as in Theorem 1 the strategy profile $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)}, \sigma_0^2)$ is an equilibrium.

To show that it is the unique equilibrium provided that R/c is high enough and λ_G^1 is low enough, we show that when the belief is equal to $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm 1 finds it dominant to exit. Notice that the continuation payoff of firm 1 at a belief of $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ is bounded above by

$$\begin{aligned} \int_0^{\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2, \lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)}{\gamma + \lambda_G^1 + \lambda_G^2}} e^{-rt}\left(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) e^{-\gamma t} \lambda_1^G R - c\right) \mathrm{d}t \\ + \mathring{\pi}^*(\lambda_G^1, \lambda_G^2) e^{-(r+\gamma)\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2, \lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)}{\gamma + \lambda_G^1 + \lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R - c}{r}, \end{aligned} \tag{12}$$

where

$$\mathring{\Omega}(p) = \frac{\gamma + (1-p)(\lambda_G^1 + \lambda_G^2)}{p}.$$

If $r > \lambda_G^1 + \lambda_G^2$,

$$e^{-(r+\gamma)\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R}{r} = \left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2, \lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R}{r}$$
$$< \left(\frac{\mathring{\Omega}(c/(\lambda_G^2 R))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R}{r}.$$

Because

$$\lim_{R\to\infty} \left(\mathring{\Omega}(c/(\lambda_G^2 R))\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R}{r}$$
$$= \lim_{c\to 0} \left(\mathring{\Omega}(c/(\lambda_G^2 R))\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2}} \frac{(\lambda_G^1 + \lambda_G^2)R}{r} = 0,$$

and since $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ can be taken to be arbitrarily close to 1 provided that λ_G^1 is sufficiently small, the expression in equation (12) negative in the limit. As a result, exiting at the cutoff belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ is dominant for firm 1 and $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)}, \sigma_0^2)$ is the unique equilibrium. □

It is easy to construct parametric examples for which $\lambda_G^2 > \lambda_G^1$, and $(\sigma_0^1, \sigma^2_{\mathring{\pi}^*(\lambda_G^2,\lambda_G^1)})$ is not an equilibrium because at the belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm one prefers to exit rather than waiting to benefit from monopoly profits. The following lemma describes the construction of the mixed strategy equilibrium we discussed in Section 3.2.

Lemma 8. *Suppose that $\lambda_G^2 > \lambda_G^1$ and $\lambda_B = 0$. If both $(\sigma_0^1, \sigma^2_{\mathring{\pi}^*(\lambda_G^2,\lambda_G^1)})$ and $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)}, \sigma_0^2)$ are equilibria, there exists a mixed strategy equilibrium such that*

(i) *firm 2 exits with probability $q > 0$ when the posterior belief is equal to $\mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$;*

(ii) *both firms exit at a positive rate when the belief belongs to $(\pi^\ddagger(\lambda_G^2+\lambda_G^1), \mathring{\pi}^*(\lambda_G^2, \lambda_G^1))$, where $\pi^\ddagger(\lambda_G^2 + \lambda_G^1) > 0$ is the belief at which a monopolist optimally exits.*

Proof. Recall that $v^i : [0,1] \to \mathbf{R}$ denotes the payoff associated with firm i's best-reply to σ_0^j. Let $W : [0,1] \to \mathbf{R}$ denote the payoff associated with the monopolist's problem.

The equilibrium we construct yields ex-ante expected payoffs equal to $(v^1(1), v^2(1))$. The probability q is chosen so that at the belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm 1 is indifferent between exiting and waiting to exit as soon as the belief falls short of $\mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$, provided that firm 2 does not exit then. That is, q satisfies

$$\int_0^{\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}} \mathring{\pi}^*(\lambda_G^1,\lambda_G^2)e^{-(\gamma+\lambda_G^1+\lambda_G^2)t}(\lambda_G^1+\lambda_G^2)\left(\frac{\lambda_1^G}{\lambda_1^G+\lambda_2^G}e^{-rt}R+e^{-rt}v^1(1) - \frac{1-e^{-rt}}{r}c\right)\mathrm{d}t$$

$$+\left(1-\frac{\lambda_G^1+\lambda_G^2}{\lambda_G^1+\lambda_G^2+\gamma}\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)\left(1-e^{-\left(\lambda_G^1+\lambda_G^2+\gamma\right)\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}}\right)\right)$$

$$\left(-\frac{c}{r}\left(1-e^{-r\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}}\right)+e^{-r\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}}qW\left(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))\right)\right)=0.$$

Clearly, exiting at some belief in $(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1),\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))$ is suboptimal. At any belief $p\in(\pi^\ddagger(\lambda_G^2+\lambda_G^1),\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))$, firm i exits at a rate

$$-\frac{p(\lambda_1^G+\lambda_2^G)\left(\frac{\lambda_G^j}{\lambda_1^G+\lambda_2^G}R+v^j(1)\right)-c}{W(p)}.$$

It can be verified that this rate is positive and bounded for any $p\in(\pi^\ddagger(\lambda_G^2+\lambda_G^1),\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))$. Consequently, along a path with no costumers, there is a strictly positive probability that both firms are still in the market by the time the belief reaches any given $p\in(\pi^\ddagger(\lambda_G^2+\lambda_G^1),\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))$. However, because the rate diverges to infinity at $\pi^\ddagger(\lambda_G^2+\lambda_G^1)$, as the denominator converges to zero, no firm will remain in the market at a belief lower than the monopoly cutoff. The rate is chosen to guarantee that each firm is indifferent between exiting, and remaining in the market and exiting in the next instant if no customer arrives. □